Linear Algebra

Problem Set 10 **2009**

Due Wednesday, 3 June 2009 at 10:00 AM in EE207. Be sure to print your name and student ID on your homework.

1. Section 6.2, Problem 10

   Suppose each number G_{k+2} is the *average* of two previous numbers G_{k+1} and G_k.

   Then $G_{k+2} = \frac{1}{2}\left(G_{k+1} + G_k\right)$:

   $$\begin{aligned} G_{k+2} &= \tfrac{1}{2}\left(G_{k+1} + G_k\right) \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find the limit as $n \to \infty$ of the matrix $A^n = S\Lambda^n S^{-1}$.

   (c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach 2/3.

2. Section 6.3, Problem 5

   A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

   $$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

   Show that the total is constant (40 people). Find the matrix in $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$?

3. When A is skew-symmetric ($A^T = -A$), $Q = e^{At}$ is orthogonal. Prove $Q^T = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^T Q = I$.

4. If $A^2 = A$, show that $e^{At} = I + (e^t - 1)A$. Use this result to compute e^{Bt}, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

5. Section 6.3, Problem 24

   Give two reasons why the matrix exponential e^{At} is never singular:

   (a) Write down its inverse.

   (b) Write down its eigenvalues. If $A\mathbf{x} = \lambda\mathbf{x}$ then $e^{At}\mathbf{x} =$ ____ $\mathbf{x}$.

6. Find an invertible matrix S and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} = SCS^{-1}$.

7. Suppose a real 3 by 3 matrix A has eigenvalues $-0.5,\ 0.2 + 0.3i,\ 0.2 - 0.3i$, with

corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1+2i \\ 4i \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1-2i \\ -4i \\ 2 \end{bmatrix}.$$

Write the general solution of $\mathbf{u}' = A\mathbf{u}$ using complex eigenvalues and eigenvectors, and then find the general *real* solution.

8. Suppose A is an n by n matrix. If $\mathbf{x}$ is in the nullspace of A, show that $M^{-1}\mathbf{x}$ is in the nullspace of $M^{-1}AM$. Is it true that $\dim N(A) = \dim N(M^{-1}AM)$?
9. Section 6.6, Problem 19

   If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. But still

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$

   (a) What sizes are the blocks of G? They are the same in each matrix.
   (b) This equation is $M^{-1}FM = G$, so F and G have the same 10 eigenvalues. F has eigenvalues of AB plus 4 zeros; G has the eigenvalues of BA plus 6 zeros. AB has the same eigenvalues as BA plus ______ zeros.
10. True or false, with a good reason:
   (a) A can't be similar to $A + I$.
   (b) If A is invertible and B is similar to A, then B is also invertible.
   (c) If A is similar to B, then A^2 is similar to B^2.
   (d) If A^2 is similar to B^2, then A is similar to B.
   (e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, the eigenvalues stay the same.