Linear Algebra
Problem Set 11 **2009**

1. Section 6.4, Problem 8
   If $A^3 = 0$ then the eigenvalues of A must be ________. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.
2. Section 6.4, Problem 16
   Even if A is rectangular, the block matrix $B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ is symmetric:
   $$B\mathbf{x} = \lambda\mathbf{x} \quad \text{is} \quad \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \lambda\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \quad \text{which is} \quad \begin{array}{l} A\mathbf{z} = \lambda\mathbf{y} \\ A^T\mathbf{y} = \lambda\mathbf{z}. \end{array}$$
   (a) Show that $-\lambda$ is also eigenvalue, with the eigenvector $(\mathbf{y}, -\mathbf{z})$.
   (b) Show that $A^T A\mathbf{z} = \lambda^2\mathbf{z}$, so that λ^2 is an eigenvalue of $A^T A$.
   (c) If $A = I$ (2 by 2) find all four eigenvalues and eigenvectors of B.
3. Section 6.4, Problem 22
   A normal matrix has $A^T A = AA^T$; it has orthogonal eigenvectors. Why is every skew-symmetric matrix normal? Why is every orthogonal matrix normal? When is $\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ normal?
4. Section 6.4, Problem 24
   Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable?
   $$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$
   Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$?
5. Section 6.4, Problem 26
   Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues?
6. Section 6.5, Problem 24
   Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding A.
7. Section 6.5, Problem 2
   For which numbers b and c are these matrices positive definite?
   $$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}.$$
   With the pivots in D and multiplier in L, factor A into LDL^T.

8. Section 6.5, Problem 20

   Give a quick reason why each of these statements is true:

   (a) Every positive definite matrix is invertible.

   (b) The only positive definite projection matrix is $P = I$.

   (c) A diagonal matrix with positive diagonal entries is positive definite.

   (d) A symmetric matrix with a positive determinant might not be positive definite!

9. Section 6.5, Problem 25

   With positive pivots in D, the factorization $A = LDL^T$ becomes $L\sqrt{D}\sqrt{D}L^T$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = L\sqrt{D}$ yields the Cholesky factorization $A = CC^T$ which is "symmetrized LU":

$$\text{From } C = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \text{ find } A. \qquad \text{From } A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \text{ find } C.$$

10. Section 6.5, Problem 28

    Without multiplying

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

    find

    (a) the determinant of A

    (b) the eigenvalues of A

    (c) the eigenvectors of A

    (d) a reason why A is symmetric positive definite.