Linear Algebra
Problem Set 3 **2009**

Due Wednesday, 25 March 2009 at 10:00 AM in EE207. Be sure to print your name and student ID on your homework.

1. An n by n matrix A is called *idempotent* if $A^2 = A$, where $A^2 = AA$.
   (a) Show that if A is idempotent and invertible, then $A = I$.
   (b) If A is idempotent, find the inverse of $I - cA$ (if possible) for some scalar c. (What is the form of the inverse? Guess it.)
2. Let E be the n by n matrix each of whose entries is 1. What is the inverse of $I - E$?
3. Section 2.4, Problem 35
   Elimination for a 2 by 2 block matrix: When you multiply the first block row by CA^{-1} and subtract from the second row, what is the "*Schur complement*" S that appears?
   $$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$
   Also, find a 2 by 2 block matrix X such that
   $$\begin{bmatrix} A & B \\ 0 & S \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} X.$$
4. With the results obtained in Problem 3, express $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as the product of three 2 by 2 matrices. And then find the inverse of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. (You only need to show it as the product of three matrices.)
5. Section 3.1, Problem 10
   Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?
   (a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.
   (b) The plane of vectors with $b_1 = 1$.
   (c) The vectors with $b_1b_2b_3 = 0$.
   (d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.
   (e) All vectors that satisfy $b_1+b_2+b_3 = 0$.
   (f) All vectors with $b_1 \leqq b_2 \leqq b_3$.
6. Section 3.1, Problem 4
   The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices.
   Write down the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$.

What matrices are in the smallest subspace containing A? (To answer this problem, you can simply show the general form of the matrices.)

7. Section 3.1, Problem 18

   True or false (check addition in each case by an example).

   (a) The symmetric matrices in **M** (with $A^T = A$) form a subspace.

   (b) The skew-symmetric matrices in **M** (with $A^T = -A$) form a subspace.

   (c) The unsymmetric matrices in **M** (with $A^T \neq A$) form a subspace.

8. What condition (equation) must be satisfied so that vector (b_1, b_2, b_3) is in the column space of A?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

9. Section 3.1, Problem 23

   If we add an extra column $\mathbf{b}$ to a matrix A, then the column space gets larger unless ________. Give an example where the column space gets larger and an example where it doesn't. Why is $A\mathbf{x} = \mathbf{b}$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$?

10. Section 3.1, Problem 27

    True or false (with a counterexample if false):

    (a) The vectors $\mathbf{b}$ that are not in the column space $C(A)$ form a subspace.

    (b) If $C(A)$ contains only the zero vector, then A is the zero matrix.

    (c) The column space of $2A$ equals the column space of A.

    (d) The column space of $A - I$ equals the column space of A.