Linear Algebra
Problem Set 8 **2009**

Due Wednesday, 13 May 2009 at 10:00 AM in EE207. Be sure to print your name and student ID on your homework.

1. Section 5.1, Problem 16
   Find the determinants of a rank one matrix and a skew-symmetric matrix:
   $$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}.$$
2. Section 5.1, Problem 18
   Use row operations to show that the 3 by 3 "Vandermonde determinant" is
   $$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$
3. Section 5.1, Problem 28
   True or false (give a reason if true or a 2 by 2 example if false):
   (a) If A is not invertible then AB is not invertible.
   (b) The determinant of A is always the product of its pivots.
   (c) The determinant of A-B equals det A – det B.
   (d) AB and BA have the same determinant.
4. Suppose P is a projection matrix and Q is an orthonormal matrix. Find all possible determinants of P and Q, respectively.
5. Section 5.1, Problem 34
   If you know that det A = 6, what is the determinant of B?
   $$\det A = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6 \qquad \det B = \begin{vmatrix} \text{row 3} + \text{row 2} + \text{row1} \\ \text{row 2} + \text{row1} \\ \text{row 1} \end{vmatrix} = ?$$
6. Section 5.2, Problem 15
   The n by n determinant C_n has 1's above and below the main diagonal:
   $$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are these determinants of C_1, C_2, C_3, C_4?

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

7. Section 5.2, Problem 27

   Block elimination subtracts CA^{-1} times the first row $[A \quad B]$ from the second row $[C \quad D]$. This leaves the *Schur complement* $D\text{-}CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

   Take determinants of these block matrices to prove correct rules for square blocks:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B| \quad \text{(if } A^{-1} \text{ exits)}$$

$$= |AD - CB| \qquad \text{(if } AC = CA\text{)}$$

8. Suppose Q is a 5 by 3 matrix with orthonormal columns. Compute the determinants of Q^TQ and QQ^T.

9. Show that if A is singular, each row of the cofactor matrix C is in the nullspace of A. Use this property to find the nullspace of A:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

10. Section 5.3, Problem 10

    From the formula $AC^T = (\det A)\, I$ show that $\det C = (\det A)^{n-1}$.