Linear Algebra

Problem Set 9 **2009**

Due Wednesday, 27 May 2009 at 10:00 AM in EE207. Be sure to print your name and student ID on your homework.

1. Section 6.1, Problem 11
   Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\mathbf{x}_2$. Any idea why this should be?
2. Section 6.1, Problem 12
   From the unit vector $\mathbf{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \mathbf{u}\mathbf{u}^T$.
   (a) Show that $P\mathbf{u} = \mathbf{u}$. Then $\mathbf{u}$ is an eigenvector with $\lambda = 1$.
   (b) If $\mathbf{v}$ is perpendicular to $\mathbf{u}$ show that $P\mathbf{v} = \mathbf{0}$. Then $\lambda = 0$.
   (c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.
3. Section 6.1, Problem 19
   A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find three of these:
   (a) the rank of B
   (b) the determinant of B^TB
   (c) the eigenvalues of B^TB
   (d) the eigenvalues of $(B+I)^{-1}$.
4. Section 6.1, Problem 27
   The block B has eigenvalues 1, 2 and C has eigenvalues 3, 4 and D has eigenvalues 5, 7. Find the eigenvalues of the 4 by 4 matrix A:
$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$
5. Section 6.1, Problem 33
   Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.
   (a) Give a basis for the nullspace and a basis for the column space.
   (b) Find a particular solution to $A\mathbf{x} = \mathbf{v}+\mathbf{w}$. Find all solutions.
   (c) Show that $A\mathbf{x} = \mathbf{u}$ has no solution. (If it did then __________would be in the column space.)
6. Section 6.1, Problem 36

There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

7. Section 6.2, Problem 20

   Find Λ and S to diagonalize $A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}$. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ________.

8. Section 6.2, Problem 25

   Show that trace(AB) = trace(BA), by adding the diagonal entries of AB and BA:

   $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

   Choose A as S and B as ΛS^{-1}. Then $S\Lambda S^{-1}$ has the same trace as $\Lambda S^{-1} S$. The trace of A equals the trace of Λ which is __________.

9. Section 6.2, Problem 30

   Suppose $A\mathbf{x} = \lambda\mathbf{x}$. If $\lambda = 0$ then $\mathbf{x}$ is in the nullspace. If $\lambda \neq 0$ then $\mathbf{x}$ is in the column space. Those spaces have dimensions $(n-r)+r=n$. So why doesn't every square matrix have n linearly independent eigenvectors?

10. Section 6.2, Problem 35

    Substituting $A = S\Lambda S^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A-\lambda I)$. The *Cayley-Hamilton theorem* says that the product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.