Linear Algebra Spring2009

Solution to Problem Set 10

1. 6.2(10)

   (a) $A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$ with $x_1 = (1,1)$, $x_2 = (1,-2)$.

   (b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-0.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

   (c) $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = A^k \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} \to \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$.

2. 6.3(5)

   $d(v+w)/dt = dv/dt + dw/dt = (w - v) + (v - w) = 0$, so the total $v + w$ is constant.

   $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = -2$ with $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

   $v(1) = 20 + 10e^{-2}$

   $w(1) = 20 - 10e^{-2}$

3.

   $Q = e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$.

   $Q^T = I + A^T t + \frac{1}{2}(A^T t)^2 + \frac{1}{6}(A^T t)^3 + \cdots = I + (-A)t + \frac{1}{2}(-At)^2 + \frac{1}{6}(-At)^3 + \cdots = e^{-At}$.

   $\therefore Q^T Q = e^{At} \cdot e^{-At} = e^0 = I$.

   $\therefore Q$ is orthogonal matrix.

4.

   $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots = I + A(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots) \quad \because A^2 = A \Rightarrow A^n = A$

   $\therefore e^{At} = I + A(e^t - 1)$.

   $\because B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

   $\therefore e^{Bt} = I + B(e^t - 1) = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$.

5. 6.3(24)

   (a) The inverse of e^{At} is e^{-At}.

(b) If $Ax = \lambda x$ then $e^{At}x = e^{\lambda t}x$ and $e^{\lambda t} \neq 0$.

6.

$\because A = SCS^{-1}$ is similar to C. They have same eigenvalues.

$\therefore tr(A) = \sum \lambda_i = tr(C)$ and $det(A) = det(C)$.

$\Rightarrow 2a = 4 \;\; a = 2$

$\Rightarrow a^2 + b^2 = 13 \;\; b = 3$

$\therefore C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$.

The eigenvalues are $\lambda_1 = 2 - 3i$, $\lambda_2 = 2 + 3i$.

for C, two eigenvector is $\begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}; C = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \Lambda \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1}$.

for A,two eigenvector is $\begin{bmatrix} 1+3i \\ 2 \end{bmatrix}, \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}; A = \begin{bmatrix} 1+3i & 1-3i \\ 2 & 2 \end{bmatrix} \Lambda \begin{bmatrix} 1+3i & 1-3i \\ 2 & 2 \end{bmatrix}^{-1}$.

$\because A = SCS^{-1} \quad \therefore \begin{bmatrix} 1+3i & 1-3i \\ 2 & 2 \end{bmatrix} = S \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$

$\Rightarrow S = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$.

7.

General solution $u(t) = \alpha e^{-0.5t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \beta e^{(0.2+0.3i)t} \begin{bmatrix} 1+2i \\ 4i \\ 2 \end{bmatrix} + \gamma e^{(0.2-0.3i)t} \begin{bmatrix} 1-2i \\ -4i \\ 2 \end{bmatrix}$.

General real solution $u(t) = \alpha e^{-0.5t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \beta' e^{0.2t} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + \beta' cos(0.3t) \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + \beta' sin(0.3t) \begin{bmatrix} -4 \\ -8 \\ 0 \end{bmatrix}$.

(即when $\beta = \gamma$,we get General real solution)

8.

$\because Ax = 0$

$\therefore M^{-1}AM(M^{-1}x) = M^{-1}Ax = 0$

so $M^{-1}x$ is in the nullspace of $M^{-1}AM$.

rank(A)=rank($M^{-1}AM$)= r $\quad \because A$ is similar to $M^{-1}AM$, have same eigenvalues.

$\therefore dim\, N(A) = dim\, N(M^{-1}AM) = n - r$.

9. 6.6(19)

(a) Diagonals 6 by 6 and 4 by 4.

(b) AB has all the same eigenvalues as BA plus 6-4=2 zeros.

10.

(a) True. $\because$the eigenvalues of (A+I) are the eigenvalues of A plus 1.

(b) True. $\because det(B) = det(A) \neq 0$

(c) True. $\because A = MBM^{-1} \Rightarrow A^2 = (MBM^{-1})(MBM^{-1}) = MB^2M^{-1}$.

(d) False. If $A = -B$, $A^2 = IB^2I^{-1}$but $A \neq IBI^{-1}$,which is not smilar.

(e) True. PAP^T is similar to A.