Linear Algebra Spring2009

Solution to Problem Set 11

1. 6.4(8)

   If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then

   $A^3 = Q\Lambda^3 Q^T = 0$ gives $\Lambda = 0$ and only symmetric possibility is $A = Q0Q^T$ =zero matrix.

2. 6.4(16)

   (a) If $Az = \lambda y$ and $A^T y = \lambda z$ then $B[y; -z] = [-Az; A^T y] = -\lambda[y; -z]$. So $-\lambda$ is also an eigenvalue of B.

   (b) $A^T Az = A^T(\lambda y) = \lambda^2 z$. The eigenvalues of $A^T A$ are≥ 0.

   (c) $\lambda = -1, -1, 1, 1$; $x_1 = (1, 0, -1, 0)$, $x_2 = (0, 1, 0, -1)$, $x_3 = (1, 0, 1, 0)$, $x_4 = (0, 1, 0, 1)$.

3. 6.4(22)

   If $A^T = -A$ then $A^T A = AA^T = -A^2$. If A is orthogonal then $A^T A = AA^T = I$.

   $A = \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ is normal only if $a = d$. Then $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

4. 6.4(24)

   A is invertible, orthogonal, permutation,diagonalizable; B is projection,diagonalizable.

   $QR, S\Lambda S^{-1}, Q\Lambda Q^T$ possible for A; $S\Lambda S^{-1}$ and $Q\Lambda Q^T$ possible for B.

5. 6.4(26)

   Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so every $\lambda = \pm 1$. Then $A = \pm I$ or

   $A = Q\Lambda Q^T = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{bmatrix}$=reflection.

6. 6.5(24)

   The sllipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $a = 1/\sqrt{\lambda_1} = \sqrt{2}$ and $b = \sqrt{2/3}$.

7. 6.5(2)

   $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^T$; Positive definite for $-3 < b < 3$.

   $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^T$; Positive definite for $c > 8$.

8. 6.5(20)

    (a) The determinant is positive, all $\lambda > 0$.

    (b) All projection matrices except I are singular.

    (c) The diagonal entries of D are its eigenvalues.

    (d) $-I$ has $det = 1$ when n is even.

9. 6.5(25)

   $A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$.

10. 6.5(28)

    $detA = 10$; $\lambda = 2$ and 5; $x_1 = (cos\theta, sin\theta)$, $x_2 = (-sin\theta, cos\theta)$; the $\lambda's$ are positive.