Linear Algebra Spring2009

Solution to Problem Set 7

1.

Let $u = \begin{bmatrix} -b \\ a \end{bmatrix}$,which is the direction of line L .

so $A = \frac{uu^T}{u^T u} = \frac{1}{a^2+b^2} \begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix}$.

2.

Let $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$.

so projection matrix $P = A(A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

and $p = Pb = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

3. 4.3(1)

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^T A\widehat{x} = A^T b$ gives $\widehat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $p = A\widehat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$.

$E = \|e\|^2 = 44$.

4. 4.3(17)

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}.$$

The solution $\widehat{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ comes from $A^T A\widehat{x} = A^T b$.

$$\Rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

5. 4.4(4)

(a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b)$(1,0)$and $(0,0)$ are orthogonal,not independent.

(c) $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right), \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right), \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$.

6.

$$rref(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow N(A) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

we want to minimize $\|x\|^2 = x^T x$

$\Rightarrow \frac{\partial x^T x}{\partial \alpha} = 0 \quad \Rightarrow \frac{\partial(10+2\alpha+6\alpha^2)}{\partial \alpha} = 0$

$\Rightarrow \alpha = -\frac{1}{6}$

$$\therefore x = \begin{bmatrix} 17/6 \\ 4/3 \\ -1/6 \end{bmatrix}$$

we can also express $x = x_A + x_{A^\perp}$, which x_A is in $R(A)$ and $x_{A^\perp}$ is in $N(A)$.

so project $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ to $N(A)$ is $\frac{1}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\therefore x = \frac{1}{6}\begin{bmatrix} 17 \\ 8 \\ -1 \end{bmatrix} + (\alpha + \tfrac{1}{6})\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad (\text{note} \begin{bmatrix} 17 \\ 8 \\ -1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix})$$

$\therefore when\ \alpha = -\frac{1}{6}$ x has minimum length $\|x\|$,and x must in the $R(A)$.

7. 4.4(15)

(a)$q_1 = \frac{1}{3}(1, 2, -2),\ q_2 = \frac{1}{3}(2, 1, 2),\ q_3 = \frac{1}{3}(2, -2, -1)$

(b) The nullspace of A^T contains q_3

(c) $\widehat{x} = (A^T A)^{-1} A^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

8. 4.4(23)

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

9. 4.4(24)

Let $A = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}^T \Rightarrow Ax = 0$

(a) x is in $N(A)$.so basis for this space is $v_1 = (-1, 1, 0, 0), \quad v_2 = (1, 0, -1, 0), \quad v_3 = (1, 0, 0, 1)$.

(b) $(1, 1, 1, -1)$.

(c) $b_2 = \frac{u^T b}{u^T u} u = \frac{1}{2}(1, 1, 1, -1)$.

$b_1 = b - b_2 = \frac{1}{2}(1, 1, 1, 3)$.

10. 4.4(34)

(a)$Qu = (I - 2uu^T)u = u - 2uu^T u$. This is $-u$, provided that $u^T u$ equals 1.

(b)$Qv = (I - 2uu^T)v = u - 2uu^T v = u$,provided that $u^T v = 0$.