Linear Algebra Spring2009

Solution to Problem Set 9

1. 6.1(11)

   Let $M = (A - \lambda_2 I)(A - \lambda_1 I)$, by Problem 6.2(35) we can prove $M = 0$ which is called "Cayley-Hamilton Theoerm". So columns of $(A - \lambda_1 I)$ is in the nullspace of $(A - \lambda_2 I)$.

   We know that $(A - \lambda_2 I)x_2 = 0$, x_2 is the null space of $(A - \lambda_2 I)$.

   Therefore columns of $(A - \lambda_1 I)$ must be multiples of the eigenvector x_2.

2. 6.1(13)

   (a) $Pu = (uu^T)u = u(u^T u) = u$ so $\lambda = 1$.

   (b) $Pv = (uu^T)v = u(u^T v) = 0$ so $\lambda = 0$.

   (c) $x_1 = (-1, 1, 0, 0)$, $x_2 = (-3, 0, 1, 0)$, $x_3 = (-5, 0, 0, 1)$ are eigenvectors with $\lambda = 0$.

3. 6.1(19)

   (a) rank=2

   (b) $det(B^T B) = 0 \quad \because det(B) = 0$

   (c) we can't find

   (d) eigenvalues of $(B + I)^{-1}$ are 1, $1/2$, $1/3$.

4. 6.1(27)

   $\lambda = 1,\ 2,\ 5,\ 7.$

5. 6.1(33)

   (a) u is a basis for the nullspace, v nad w give a basis for the column space.

   (b) $x_p = \left(0, \frac{1}{3}, \frac{1}{5}\right)$ is a particular solution. $x = x_p + \alpha u$.

   (c) If $Ax = u$ had a solution, u would be in the column space, giving demention 3.

6. 6.1(36)

   For 3 by 3 permutations: determinant = 1 or -1, all pivots =1, trace = 0,1 or 3, eigenvalues = 1 or -1 or $e^{2\pi i/3}$ or $e^{4\pi i/3}$.

7. 6.2(20)

   $$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};\ \Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } S\Lambda^k S^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{: steady state.}$$

8. 6.2(25)

   trace $AB = (aq + bs) + (cr + dt) = (qa + rc) + (sb + td)$ =traceBA.

   Proof for diagonalizable case: the trace of $S\Lambda S^{-1}$ is the trace of $(\Lambda S^{-1})S = \Lambda$ which is the sum of the $\lambda's$.

9. 6.2(30)

   Two problems:

   1.) The nullspace and column space can overlap, so x could be in both.

   2.) There may not be r independent eigenvectors in the column space.

10. 6.2(35)

    If $A = S\Lambda S^{-1}$ then the product $(A - \lambda_1 I)\cdots(A - \lambda_n I)$ equals $S(\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)S^{-1}$.

    The factor $(\Lambda - \lambda_j I)$ is zero in row j. The product is zero in all rows = zero matrix.