Linear Algebra

Problem Set 11 **2010**

Due Wednesday, 2 June 2010 at 10:00 AM in EE102. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (20pts) Section 6.3, Problem 4

   A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

   $$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

   Show that the total $v + w$ is constant (40 people). Find the matrix in $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and at $t = \infty$?

2. (15pts) When A is skew-symmetric ($A^T = -A$), $Q = e^{At}$ is orthogonal. Prove $Q^T = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^TQ = I$.

3. (15pts) Find an invertible matrix S and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = \begin{bmatrix} 17 & 26 \\ -10 & -15 \end{bmatrix} = SCS^{-1}$.

4. (15pts) Suppose a real 3 by 3 matrix A has eigenvalues $-0.5,\ 0.2+0.3i,\ 0.2-0.3i,$ with corresponding eigenvectors

   $$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1-2i \\ 4i \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1+2i \\ -4i \\ 2 \end{bmatrix}.$$

   Write the general solution of $\mathbf{u}' = A\mathbf{u}$ using complex eigenvalues and eigenvectors, and then find the general *real* solution.

5. (20pts) True or false, with a good reason:
   (a) A can't be similar to $A + I$.
   (b) If A is invertible and B is similar to A, then B is also invertible.
   (c) If A is similar to B, then A^3 is similar to B^3.
   (d) If A^2 is similar to B^2, then A is similar to B.
   (e) It is impossible that A is similar to A^{-1}.

6. (15pts) If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. But with blocks

$$M^{-1}FM = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$

(a) What sizes are the four blocks (the same four sizes in each matrix)?

(b) This equation is $M^{-1}FM = G$, so F and G have the same 10 eigenvalues. F has the 6 eigenvalues of AB plus 4 zeros; G has the 4 eigenvalues of BA plus 6 zeros. Show that AB has the same eigenvalues as BA plus k zeros. What is k?