Linear Algebra

Problem Set 7 **2010**

Due Wednesday, 28 April 2010 at 10:00 AM in EE102. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (25pts) Consider two bases $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ and $\mathcal{C}=\left\{\begin{bmatrix}5\\4\\5\end{bmatrix},\begin{bmatrix}-5\\-6\\-5\end{bmatrix}\right\}$ for a vector space *V*. Consider a linear transform $T:V\to V$, defined by

   $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}3\\2\\3\end{bmatrix}, T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}2\\0\\2\end{bmatrix}$. Suppose the coordinate vector of **x** with respect to basis $\mathcal{B}$ is $[\mathbf{x}]_B=\begin{bmatrix}3\\1\end{bmatrix}$.

   (a) Find $T(\mathbf{x})$.

   (b) Find $[\mathbf{x}]_C$ and $[T(\mathbf{x})]_C$.

   (c) Find the change-of-coordinates matrix from basis $\mathcal{B}$ to $\mathcal{C}$, $[B]_C$. Note that

   $[\mathbf{x}]_C=[B]_C[\mathbf{x}]_B$.

   (d) Find the 2 by 2 matrix representation for *T* with respect to basis $\mathcal{B}$.

   (e) Find the 2 by 2 matrix representation for *T* with respect to basis $\mathcal{C}$..

2. (15pts) Let $\mathcal{B}=\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$ and $\mathcal{C}=\{\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3\}$ be bases for a vector space *V*, and suppose $\mathbf{b}_1=4\mathbf{c}_1-\mathbf{c}_2$, $\mathbf{b}_2=-\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3$, $\mathbf{b}_3=\mathbf{c}_2-2\mathbf{c}_3$.

   (a) Find the 2nd column of the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

   (b) Find the 3rd column of the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

3. (25pts) Suppose *W* is the subspace spanned by $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-5\\1\\3\end{bmatrix}$.

(a) Find the point in W that is closest to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) Find the projection matrix P_W onto W. What is $\text{rank}P_W$?

(c) Find the projection matrix $P_{W^\perp}$ onto $W^\perp$. What is $\text{rank}P_{W^\perp}$?

(d) Find a basis for $W^\perp$. (Think of the column space of $P_{W^\perp}$.)

4. (15pts) Show that if $P^T = P$ and $P^2 = P$ then P is an orthogonal projection matrix. Note that P is an orthogonal projection matrix if $(I-P)\mathbf{x} \perp P\mathbf{y}$, for any **x** and **y**. This means that the column space of P is orthogonal to the column space of $(I-P)$.

5. (20pts) Section 4.2, Problem 30

(a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$\begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. Multiply $B=P_CAP_R$. Your answer B should be a little surprising—can you explain it?