HW 1 solution

1.

$f(t) = a_3t^3 + a_2t^2 + a_1t + a_0$

$f(2) = 9 = 8a_3 + 4a_2 + 2a_1 + a_0 \quad ---(1)$

$f(3) = 32 = 27a_3 + 9a_2 + 3a_1 + a_0 \quad ---(2)$

$f^{'}(t) = 3a_3t^2 + 2a_2t + a_1$

$f^{'}(1) = 0 = 3a_3 + 2a_2 + a_1 \quad ---(3)$

$f^{'}(2) = 12 = 12a_3 + 4a_2 + a_1 \quad ---(4)$

From (1)(2)(3)(4)

$$=> \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 12 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 32 \\ 0 \\ 12 \end{bmatrix}$$

$\therefore$ The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 9 \\ 1 & 3 & 9 & 27 & 32 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 12 & 12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 9 \\ 0 & 1 & 5 & 19 & 23 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 12 & 12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 9 \\ 0 & 1 & 5 & 19 & 23 \\ 0 & 0 & -3 & -16 & -23 \\ 0 & 0 & -1 & -7 & -11 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 9 \\ 0 & 1 & 5 & 19 & 23 \\ 0 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 5 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

=> $a_0 = 5, a_1 = 0, a_2 = -3, a_3 = 2$

2.

Use Gaussian elimination

$$\Rightarrow [A \mid b] = \left[\begin{array}{ccc|c} 2 & -2 & 1 & 1 \\ 4 & -3 & -1 & -2 \\ -6 & 7 & -6 & 2t-1 \end{array}\right] \xrightarrow[R_{13}(3)]{R_{12}(-2)} \left[\begin{array}{ccc|c} 2 & -2 & 1 & 1 \\ 0 & 1 & -3 & -4 \\ 0 & 1 & -3 & 2t+2 \end{array}\right] \xrightarrow[R_{23}(-1)]{R_{21}(2)} \left[\begin{array}{ccc|c} 2 & 0 & -5 & 1 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 2t+6 \end{array}\right]$$

$\therefore$When t=-3，AX=b has infinite solutions.

3.

Apply the following operations to *I* to obtain *P*

a. Add row1 to row2
b. Add (-1)*row2 to row1
c. Add row1 to row2
d. Multiply row1 by (-1)

$$\mathbf{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{a}\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}\xrightarrow{b}\begin{bmatrix}0&-1&0\\1&1&0\\0&0&1\end{bmatrix}\xrightarrow{c}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\xrightarrow{d}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\mathbf{P}$$

4.

According to the question，the elimination matrices multiply at the left of K.

So we know that the Gaussian elimination is row operation.

The first three steps are following

$$\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\xrightarrow{R_{12}(\frac{1}{2})}\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\xrightarrow{R_{23}(\frac{2}{3})}\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&1&0\\0&0&\frac{4}{3}&-1\\0&0&-1&2\end{bmatrix}\xrightarrow{R_{34}(\frac{3}{4})}\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&0&\frac{5}{4}\end{bmatrix}$$

$$\therefore E_{43}E_{32}E_{21}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{3}{4}&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\frac{2}{3}&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

Use the same Gaussian elimination on the identity matrix.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{R_{12}(\frac{1}{2})}\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{R_{23}(\frac{2}{3})}\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\\frac{1}{3}&\frac{2}{3}&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{R_{34}(\frac{3}{4})}\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\\frac{1}{3}&\frac{2}{3}&1&0\\\frac{1}{4}&\frac{1}{2}&\frac{3}{4}&1\end{bmatrix}$$

We find that the result is equal to $E_{43}E_{32}E_{21}$.

5.

$$[\mathbf{A} \mid \mathbf{b}_1 \mid \mathbf{b}_2 \mid \mathbf{b}_3] \sim \left[\begin{array}{ccc|c|c|c} 4 & -8 & 5 & 1 & 0 & 0 \\ 4 & -7 & 4 & 0 & 1 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & -2 & 5/4 & 1/4 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -7/4 & 5/4 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c|c|c} 1 & -2 & 5/4 & 1/4 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1/4 & 5/4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & -6 & 10 & -5 \\ 0 & 1 & 0 & 4 & -7 & 4 \\ 0 & 0 & 1 & 5 & -8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & -4 & 3 \\ 0 & 1 & 0 & 4 & -7 & 4 \\ 0 & 0 & 1 & 5 & -8 & 4 \end{array}\right]$$

Now we have 3 sets of solutions corresponding to $[\mathbf{A}|\mathbf{b}_1]$, $[\mathbf{A}|\mathbf{b}_2]$ and $[\mathbf{A}|\mathbf{b}_3]$ respectively.

$$=> \begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ -7 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Combine the above equations.

$$=> \begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix}$$

6.

(a)

$\because \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}$ is 4x3 and $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 5 & 3 \\ 3 & 5 & 4 \\ 4 & 5 & 5 \end{bmatrix}$ is also 4x3.

$\therefore$X is 3×3.

$\Rightarrow$Let X= $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=x_{11}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+x_{21}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+x_{31}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\Rightarrow x_{11}=0\ \ x_{21}=1\ \ x_{31}=0$$

$$\Rightarrow\begin{bmatrix}5\\5\\5\\5\end{bmatrix}=x_{12}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+x_{22}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+x_{32}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\Rightarrow x_{12}=0\ \ x_{22}=0\ \ x_{32}=5$$

$$\begin{bmatrix}3\\3\\4\\5\end{bmatrix}=x_{13}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+x_{23}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+x_{33}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\Rightarrow x_{13}=1\ \ x_{23}=1\ \ x_{33}=1$$

$$\therefore \mathrm{X}=\begin{bmatrix}0&0&1\\1&0&1\\0&5&1\end{bmatrix}.$$

(b)

$\because \begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$ is 4×4 and $\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$ is also 4×4. $\therefore$X is 4×4.

$\Rightarrow$Let X= $\begin{bmatrix}x_{11}&x_{12}&x_{13}&x_{14}\\x_{21}&x_{22}&x_{23}&x_{24}\\x_{31}&x_{32}&x_{33}&x_{34}\\x_{41}&x_{42}&x_{43}&x_{44}\end{bmatrix}$

$\Rightarrow$

$$\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} = x_{11}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + x_{12}\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix} + x_{13}\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix} + x_{14}\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_{11} & x_{12} & x_{13} & x_{14}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix} = x_{21}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + x_{22}\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix} + x_{23}\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix} + x_{24}\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_{21} & x_{22} & x_{23} & x_{24}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix} = x_{31}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + x_{32}\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix} + x_{33}\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix} + x_{34}\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_{31} & x_{32} & x_{33} & x_{34}\end{bmatrix} = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} = x_{41}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + x_{42}\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix} + x_{43}\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix} + x_{44}\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_{41} & x_{42} & x_{43} & x_{44}\end{bmatrix} = \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix}0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0\end{bmatrix}$$