Solutions to Homework 10

1.

$$\mathbf{A}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\sim\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad\therefore\text{rank}(\mathbf{A})=1$$

Consider the equation $\mathbf{Ax}=0=0\cdot\mathbf{x}$

$\Rightarrow$ nullity($\mathbf{A}$) = number of zero eigenvalues of $\mathbf{A}$

$\Rightarrow\ \lambda_1=\lambda_2=\lambda_3=0$

Also, $\sum_{i=1}^{4}\lambda_i=tr(\mathbf{A})\quad\therefore\lambda_4=4$

$$\mathbf{B}=\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad\therefore\text{rank}(\mathbf{B})=2$$

$\Rightarrow$ Expect that $\lambda_1=\lambda_2=0$

$$\det(\mathbf{B}\text{-}\lambda\mathbf{I})=\begin{vmatrix}1-\lambda&0&1&0\\0&1-\lambda&0&1\\1&0&1-\lambda&0\\0&1&0&1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}1-\lambda&0&1\\0&1-\lambda&0\\1&0&1-\lambda\end{vmatrix}+1\begin{vmatrix}0&1&0\\1-\lambda&0&1\\1&0&1-\lambda\end{vmatrix}$$

$=\lambda^4-4\lambda^3+4\lambda^2=0$

$\Rightarrow\lambda^2(\lambda^2-4\lambda+4)=0$

$\Rightarrow\lambda_1=\lambda_2=0,\lambda_3=\lambda_4=2$

Another approach:

Observe that $\mathbf{B}^2=2\mathbf{B}$

$\Rightarrow\lambda^2-2\lambda=0$

$\Rightarrow\lambda=0,2$

$\because$ nullity($\mathbf{B}$)=2

$\therefore$ $\mathbf{B}$ has two zero eigenvalues

$\because\sum_{i=1}^{4}\lambda_i=tr(\mathbf{B})$

$\Rightarrow\lambda=0,0,2,2$

2.

Use characteristic polynomial to prove it

$\Rightarrow (\mathbf{A}-\lambda\mathbf{I}) = [(\mathbf{A}^{\mathrm{T}}-\lambda\mathbf{I})]^{\mathrm{T}}$

$\Rightarrow \left|(\mathbf{A}-\lambda\mathbf{I})\right| = \left|[(\mathbf{A}^{\mathrm{T}}-\lambda\mathbf{I})]^{\mathrm{T}}\right| = \left|(\mathbf{A}^{\mathrm{T}}-\lambda\mathbf{I})\right| = 0$

The characteristic polynomials of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ are the same ,
so we can use them to find the same eigenvalues.

For example that the eigenvectors of A and AT are not the same:

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}, \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2 & 1 & 1 \\ -3 & -2 & -3 \\ 1 & 1 & 2 \end{bmatrix}$$

They have the same eigenvalues which are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$, but the eigenvectors are different.

Eigenvectors of A are $\left\{ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}} , \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^{\mathrm{T}} , \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}} \right\}$

Eigenvectors of A^{T} are

$\left\{ \begin{bmatrix} -0.3015 & 0.9045 & -0.3015 \end{bmatrix}^{\mathrm{T}} , \begin{bmatrix} -0.1090 & -0.6463 & 0.7553 \end{bmatrix}^{\mathrm{T}} , \begin{bmatrix} 0.3736 & -0.8155 & 0.4420 \end{bmatrix}^{\mathrm{T}} \right\}$

3.

(a)

$\mathbf{A}$ has one zero eigenvalue

$\therefore$ nullity($\mathbf{A}$)=1

$\therefore$ rank($\mathbf{A}$)=3-1=2

(b)

$\det(\mathbf{A}^T\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{A}) = \det(\mathbf{A}^T)\cdot(-1)\cdot 0\cdot 1 = 0$

(c)

$\mathbf{A}^T\mathbf{A}$ is 3 by 3.

$\text{rank}(\mathbf{A}^T\mathbf{A}) = \text{rank}(\mathbf{A})=2$

$\therefore$ One eigenvalue is 0. The other two are unknown.

(d)

Let λ be the eigenvalue of $\mathbf{A}$

$\Rightarrow \mathbf{Ax} = \lambda\mathbf{x}$

$\Rightarrow \mathbf{A}^2\mathbf{x} = \mathbf{AAx} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{Ax} = \lambda^2\mathbf{x}$

$\Rightarrow$ Eigenvalue of $\mathbf{A^2}$ is $\lambda^2 ---(1)$

Let λ be the eigenvalue of $\mathbf{A}$

$\Rightarrow \mathbf{Ax} = \lambda\mathbf{x}$

$\Rightarrow \mathbf{Ax} + \mathbf{x} = \lambda\mathbf{x} + \mathbf{x}$

$\Rightarrow \mathbf{(A + I)x = (}\lambda + 1\mathbf{)x}$

$\Rightarrow$ Eigenvalue of $\mathbf{A + I}$ is $\lambda + 1 ---(2)$

Let λ be the eigenvalue of $\mathbf{A}$

$\Rightarrow \mathbf{Ax} = \lambda\mathbf{x}$

$\Rightarrow \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}(\lambda\mathbf{x})$

$\Rightarrow \mathbf{x}=\lambda\mathbf{A}^{-1}\mathbf{x}$

$\Rightarrow \mathrm{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$

$\Rightarrow$ Eigenvalue of $\mathbf{A}^{-1}$ is $\lambda^{-1} ---(3)$

Based on (1)(2)(3)

$\Rightarrow$ Answer = 0.5, 1, 0.5

4.

(a)

$\mathbf{Ax} = \lambda \mathbf{x}$

$\Rightarrow (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$

$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = 0$

We can find the eigenvalues which are $\lambda_1 = 1$ and $\lambda_2 = c - 0.6$

And , the eigenvectors are $\mathbf{x}_1 = [1 - c \quad 0.6]^{\mathrm{T}}$ and $\mathbf{x}_2 = [1 \quad -1]^{\mathrm{T}}$

(b)

When $c = 1.6$, A=$\begin{bmatrix} 0.4 & -0.6 \\ 0.6 & 1.6 \end{bmatrix}$, and $\lambda_1 = \lambda_2 = 1$

$\Rightarrow (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$

$\Rightarrow \begin{bmatrix} -0.6 & -0.6 \\ 0.6 & 0.6 \end{bmatrix} \mathbf{x} = 0$

$\Rightarrow \mathbf{x} = [1 \quad -1]^{\mathrm{T}}$

$\therefore$ all solutions to $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$ are multiples of $\mathbf{x} = [1 \quad -1]^{\mathrm{T}}$

(c)

When $c = 0.8$, A=$\begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$, and $\lambda_1 = 1$ and $\lambda_2 = 0.2$

We diagonalize $\mathbf{A}$ first first

$\Rightarrow \mathbf{A} = \mathbf{S\Lambda S}^{-1}$

$\because \mathbf{A}^{\mathrm{n}} = \mathbf{S\Lambda}^{\mathrm{n}}\mathbf{S}^{-1}$

$\Rightarrow \lim_{n\to\infty} \mathbf{A}^{\mathrm{n}} = \lim_{n\to\infty} \mathbf{S\Lambda}^{\mathrm{n}}\mathbf{S}^{-1} = \lim_{n\to\infty} \mathbf{S} \begin{bmatrix} 1^{\mathrm{n}} & 0 \\ 0 & 0.2^{\mathrm{n}} \end{bmatrix} \mathbf{S}^{-1} \simeq \mathbf{S} \begin{bmatrix} 1^{\mathrm{n}} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{S}^{-1}$

$[1 \quad 3]^{\mathrm{T}}$ is the eigenvector corresponding to $\lambda_1 = 1$

$[1 \quad -1]^{\mathrm{T}}$ is the eigenvector corresponding to $\lambda_2 = 0.2$

$\therefore \mathbf{A}^{\infty} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

5.

(a)

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

Eigenvalues and eigenvectors of $\mathbf{A}$ are 1, -0.5 and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b)

$$\mathbf{A} = \mathbf{S\Lambda S}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$\lim_{n\to\infty} \mathbf{A}^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \lim\limits_{n\to\infty} 1^n & 0 \\ 0 & \lim\limits_{n\to\infty} (-0.5)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

(c)

$$\lim_{k\to\infty} \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \lim_{n\to\infty} \mathbf{A}^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$

6.

Substitute 1 $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$ into $(\mathbf{A} - \lambda\mathbf{I})$

$\Rightarrow (\mathbf{A} - \lambda\mathbf{I}) = (\mathbf{S\Lambda S}^{-1} - \lambda\mathbf{I}) = \mathbf{S}(\mathbf{\Lambda} - \lambda\mathbf{I})\mathbf{S}^{-1}$

$\Rightarrow (\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I})(\mathbf{A} - \lambda_3\mathbf{I}) \cdots (\mathbf{A} - \lambda_n\mathbf{I})$

$$= \mathbf{S} \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \mathbf{S}^{-1}\mathbf{S} \begin{bmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \mathbf{S}^{-1} \cdots \mathbf{S} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \mathbf{S}^{-1}$$

$$= \mathbf{S} \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \cdots \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \mathbf{S}^{-1}$$

$= 0$

Then $p(\mathbf{A})$ = zero matrix , which is the Cayley-Hamilton Theorem.

(If A is not diagonalizable , one proof is to take a sequence of diagonalizable matrices approaching A.)