Solutions to Homework 11

1.

$$\frac{d(v+w)}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = (w-v) + (v-w) = 0$$

$\therefore v + w$ is constant.

$$\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$$

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \mathbf{Au}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\Rightarrow \lambda_1 = -2, \lambda_2 = 0, \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{u}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{0t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{u}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

$$\Rightarrow c_1 = 10, c_2 = 20$$

$$\Rightarrow \mathbf{u}(t) = 10e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \mathbf{u}(1) = \begin{bmatrix} v(1) \\ w(1) \end{bmatrix} = 10e^{-2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}(\infty) = \begin{bmatrix} v(\infty) \\ w(\infty) \end{bmatrix} = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2.

Use Taylor series method

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \frac{1}{4!}(\mathbf{A}t)^4 + \frac{1}{5!}(\mathbf{A}t)^5 + \cdots\cdots$$

$\Rightarrow$

$$(e^{\mathbf{A}t})^{\mathrm{T}} = \mathbf{I} + (\mathbf{A}^{\mathrm{T}}t) + \frac{1}{2!}(\mathbf{A}^{\mathrm{T}}t)^2 + \frac{1}{3!}(\mathbf{A}^{\mathrm{T}}t)^3 + \frac{1}{4!}(\mathbf{A}^{\mathrm{T}}t)^4 + \frac{1}{5!}(\mathbf{A}^{\mathrm{T}}t)^5 + \cdots\cdots = e^{\mathbf{A}^{\mathrm{T}}t} = e^{-\mathbf{A}t}$$

If $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ are eigenvalues of $\mathbf{A}$ and $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$

$$\Rightarrow e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{\Lambda}}\mathbf{S}^{-1}, \quad e^{-\mathbf{A}t} = \mathbf{S}e^{-\mathbf{\Lambda}}\mathbf{S}^{-1}$$

$$\Rightarrow \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = e^{-\mathbf{A}t}e^{\mathbf{A}t} = \mathbf{S}e^{-\mathbf{\Lambda}}\mathbf{S}^{-1}\mathbf{S}e^{\mathbf{\Lambda}}\mathbf{S}^{-1} = \mathbf{S}\begin{bmatrix} e^{-\lambda_1} & & \\ & \ddots & \\ & & e^{-\lambda_n} \end{bmatrix}\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}\mathbf{S}^{-1} = \mathbf{SIS}^{-1} = \mathbf{I}$$

3.

$$\mathbf{A} = \begin{bmatrix} 17 & 26 \\ -10 & -15 \end{bmatrix}$$

$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$\Rightarrow \lambda^2 - 2\lambda + 5 = 0$

$\Rightarrow \lambda = 1 + 2i, 1 - 2i$

$(\mathbf{A} - (1-2i)\mathbf{I})\mathbf{x} = 0$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} -8+i \\ 5 \end{bmatrix}$$

$$\Rightarrow \mathbf{A} = \begin{bmatrix} -8 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -8 & 1 \\ 5 & 0 \end{bmatrix}^{-1}$$

4.

$\mathbf{u}' = \mathbf{Au}$

Assume $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$, let $\mathbf{u} = \mathbf{Sv}$

$\Rightarrow \mathbf{Sv}' = \mathbf{ASv} = \mathbf{S\Lambda S}^{-1}\mathbf{Sv} = \mathbf{S\Lambda v}$, mutiply $\mathbf{S}^{-1}$ with both sides

$$\Rightarrow \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\Rightarrow v_1 = c_1 e^{\lambda_1 t}$, $v_2 = c_2 e^{\lambda_2 t}$, $v_3 = c_3 e^{\lambda_3 t}$ where c_1 , c_2 and c_3 are any constant

$$\Rightarrow \mathbf{u} = \mathbf{Sv} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1-2i \\ 4i \\ 2 \end{bmatrix} e^{(0.2+0.3i)t} + c_3 \begin{bmatrix} 1+2i \\ -4i \\ 2 \end{bmatrix} e^{(0.2-0.3i)t}$$

$\therefore$ The general real solution is $\mathrm{Re}\{\mathbf{u}\}$

$$= c_1 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 e^{0.2t} \begin{bmatrix} \cos 0.3t + 2\sin 0.3t \\ -4\sin 0.3t \\ 2\cos 0.3t \end{bmatrix} + c_3 e^{0.2t} \begin{bmatrix} \cos 0.3t + 2\sin 0.3t \\ -4\sin 0.3t \\ 2\cos 0.3t \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 e^{0.2t} \begin{bmatrix} \cos 0.3t + 2\sin 0.3t \\ -4\sin 0.3t \\ 2\cos 0.3t \end{bmatrix} \text{, let } c_2(\text{new}) = c_2(\text{old}) + c_3$$

5.

(a)

True

Because eigenvalues of $\mathbf{A}+\mathbf{I}$ are eigenvalues of $\mathbf{A}$ plus one.

They have different eigenvalues so they're not similar.

(b)

True

If $\mathbf{B}$ is similar to $\mathbf{A}$ $\Rightarrow$ $\mathbf{B} = \mathbf{MAM}^{-1}$

$\mathbf{B}^{-1} = (\mathbf{MAM}^{-1})^{-1} = \mathbf{MA}^{-1}\mathbf{M}^{-1}$ exists. ($\because$ $\mathbf{A}$ is invertible)

(c)

True

If $\mathbf{A}$ is similar to $\mathbf{B}$ $\Rightarrow$ $\mathbf{A} = \mathbf{MBM}^{-1}$

$\mathbf{A}^3 = (\mathbf{MBM}^{-1})(\mathbf{MBM}^{-1})(\mathbf{MBM}^{-1}) = \mathbf{MB}^3\mathbf{M}^{-1}$

$\therefore$ $\mathbf{A}^3$ is similar to $\mathbf{B}^3$.

(d)

False

Let $\mathbf{A}^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \mathbf{B}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

They are similar because they have the same eigenvalues 2,3.

But one choice of $\mathbf{A} = \begin{bmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$

They are not similar because they have different eigenvalues.

(e)

False

If $\mathbf{A} = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{I}$

They are similar.

6.

(a)

The sizes of four block matrices in each matrix are $\begin{bmatrix} 6\times 6 & 6\times 4 \\ 4\times 6 & 4\times 4 \end{bmatrix}$

(b)

Assume $\{\lambda_1 , \lambda_2 , \lambda_3 , \lambda_4 , \lambda_5 , \lambda_6\}$ are eigenvalues of **AB** and $\{\lambda_7 , \lambda_8 , \lambda_9 , \lambda_{10}\}$ are eigenvalues of **BA**

F and **G** have the same 10 eigenvalues.**F** has the 6 eigenvalues of **AB** plus 4 zeros;
G has the 4 eigenvalues of **BA** plus 6 zeros.

$\Rightarrow$

Eigenvalues of **F** are $\{\lambda_1 , \lambda_2 , \lambda_3 , \lambda_4 , \lambda_5 , \lambda_6 , 0 , 0 , 0 , 0\}$,
Eigenvalues of **G** are $\{\lambda_7 , \lambda_8 , \lambda_9 , \lambda_{10} , 0 , 0 , 0 , 0 , 0 , 0\}$

$\Rightarrow$

The eigenvalues of **BA** will be equal to the four eigenvalues of **AB**.
And we know that tr(**AB**)=tr(**BA**) = summation of its eigenvalues.
$\Rightarrow \mathrm{tr}(\mathbf{AB}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = \mathrm{tr}(\mathbf{BA}) = \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10}$
$\therefore$ **AB** must have the same eigenvalues as **BA** plus 2 zeros.