Solutions to Homework 12

1.

(a) False

$$\mathbf{A}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \lambda=1,1, \mathbf{x}=\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ 0\end{bmatrix}$$

(b) True

Suppose $\mathbf{Ax}_1=\lambda_1\mathbf{x}_1, \mathbf{Ax}_2=\lambda_2\mathbf{x}_2$

$\Rightarrow \mathbf{x_2}^{\mathbf{T}}\mathbf{Ax_1} = \lambda_1\mathbf{x_2}^{\mathbf{T}}\mathbf{x_1}=0$

$\mathbf{x}_1^{\mathbf{T}}\mathbf{Ax}_2 = \lambda_2\mathbf{x}_1^{\mathbf{T}}\mathbf{x}_2=0$ (orthogonal eigenvectors)

$\Rightarrow (\mathbf{x_2}^{\mathbf{T}}\mathbf{Ax_1})^{\mathrm{T}}=0=\mathbf{x}_1^{\mathbf{T}}\mathbf{Ax}_2$

$\Rightarrow \mathbf{x}_1^{\mathbf{T}}\mathbf{A}^{\mathrm{T}}\mathbf{x}_2=\mathbf{x}_1^{\mathbf{T}}\mathbf{Ax}_2$

$\Rightarrow \mathbf{A}=\mathbf{A}^{\mathrm{T}}$

(c) True

Symmetric matrices are orthogonally diagonalizable.

$\mathbf{A} = \mathbf{Q\Lambda Q}^{-1}$, where $\mathbf{Q}$ is an orthogonal matrix, i.e., $\mathbf{Q}^{\mathrm{T}}=\mathbf{Q}^{-1}$

$\Rightarrow \mathbf{A}^{-1} = (\mathbf{Q\Lambda Q}^{-1})^{-1}=\mathbf{Q\Lambda Q}^{-1}$

$(\mathbf{A}^{-1})^{\mathrm{T}}=(\mathbf{Q\Lambda Q}^{-1})^{\mathrm{T}}=\mathbf{Q\Lambda Q}^{\mathrm{T}}=\mathbf{Q\Lambda Q}^{-1}=\mathbf{A}^{-1}$

(d) False

$$\mathbf{A}=\begin{bmatrix}2 & -2 & 0\\ -2 & -1 & 3\\ 0 & 3 & 3\end{bmatrix}$$

$\Rightarrow \lambda=-3.2151,\ 2.2595,\ 4.9556$

$$\mathbf{S}=\begin{bmatrix}0.3265 & 0.8794 & 0.3466\\ 0.8512 & -0.1141 & -0.5122\\ -0.4109 & 0.4623 & -0.7858\end{bmatrix}$$ (Not symmetric.)

2

$\det(\mathbf{A}) \neq 0 \Rightarrow$ invertible

$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I} \Rightarrow$ orthogonal

$\mathbf{A}^2 \neq \mathbf{A} \Rightarrow \mathbf{A}$ isn't a projection matrix

$\mathbf{A}$ is obtained by exchaging rows of the identity matrix $\Rightarrow$ permutation

$\mathbf{A}$ is real symmetric $\Rightarrow \mathbf{A}$ can be orthogonal diagonalized , $\mathbf{A} = \mathbf{S\Lambda S}^{-1} = \mathbf{Q\Lambda Q}^{\mathrm{T}}$

$\mathbf{A}$ is the permutation matrix $\Rightarrow \mathbf{A}$ doesn't exsist LU factorization

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{QR} \Rightarrow \text{A exsists QR factorization}$$

$\det(\mathbf{B}) = 0 \Rightarrow \mathbf{B}$ isn't invertible

$\mathbf{B}^{\mathrm{T}}\mathbf{B} \neq \mathbf{I} \Rightarrow \mathbf{B}$ isn't orthogonal

$\mathbf{B}^2 = \mathbf{B} \Rightarrow$ projection

$\mathbf{B}$ isn't obtained by exchaging rows of the identity matrix $\Rightarrow \mathbf{B}$ isn't a permutation matrix

$\mathbf{B}$ is real symmetric $\Rightarrow \mathbf{B}$ can be orthogonal diagonalized , $\mathbf{B} = \mathbf{S\Lambda S}^{-1} = \mathbf{Q\Lambda Q}^{\mathrm{T}}$

Rank of $\mathbf{B}$ is $1 \Rightarrow \mathbf{B}$ doesn't exsist LU factorization

$\mathbf{B}$ doesn't have independent columns $\Rightarrow \mathbf{B}$ doesn't exsist QR factorization

3.

(a)

$\mathbf{x^T Ax = (Ax)^T x = x^T A^T x = -x^T Ax}$

$\Rightarrow 2\mathbf{x^T Ax} = 0$

$\Rightarrow \mathbf{x^T Ax} = 0$

(b)

Let $\mathbf{z}$ be an eigenvector of $\mathbf{A}$, assume $\mathbf{z}=\mathbf{x}+i\mathbf{y}$, $\mathbf{x}, \mathbf{y} \in R$

$\bar{\mathbf{z}}^{\mathbf{T}}\mathbf{Az} = \lambda \bar{\mathbf{z}}^{\mathbf{T}}\mathbf{z} = \lambda \|\mathbf{z}\|^2$ ---(1)

Also,

$\bar{\mathbf{z}}^{\mathbf{T}}\mathbf{Az} = (\mathbf{x} - \mathrm{i}\mathbf{y})^T \mathbf{A}(\mathbf{x} + i\mathbf{y}) = \mathbf{x^T Ax} + i\mathbf{x^T Ay} - i\mathbf{y^T Ax} + \mathbf{y^T Ay} = i(\mathbf{x^T Ay} - \mathbf{y^T Ax})$ ---(2)

$(1) = (2)$

$\Rightarrow \lambda$ is pure imaginary.

(c)

λ's are the roots of det($\mathbf{A}$-$\lambda\mathbf{I}$)=0

$\therefore$ pairs of λ: ib,-ib

$\Rightarrow \det(\mathbf{A}) = \lambda_1 \cdots \lambda_n \geq 0$

4.

det $\mathbf{A}_k > 0 \Rightarrow \mathbf{A}$ is positive definite

det $\mathbf{A}_1 = a > 0$

det $\mathbf{A}_2 = a^2 - 1 = (a+1)(a-1) > 0 \Rightarrow a > 1$ or $a < -1$

det $\mathbf{A}_3 = a^3 - 2a + 2 > 0 \Rightarrow a > -1.7$

$\therefore$ If $a > 1$, $\mathbf{A}$ is positive definite

det $\mathbf{B}_k > 0 \Rightarrow \mathbf{B}$ is positive definite

det $\mathbf{B}_1 = 1 > 0$

det $\mathbf{B}_2 = b - 4 > 0 \Rightarrow b > 4$

det $\mathbf{B}_3 = -4b + 12 > 0 \Rightarrow b < 3$

$\Rightarrow b > 4 \rightarrow\leftarrow b < 3$

$\therefore$ It is impossible that $\mathbf{B}$ is positive definite

5.

(a)

All $\lambda_1 > 0$

$\therefore$ det(**A**)=$\lambda_1 \cdots \lambda_n \neq 0$

$\therefore$ **A** is non-singular.

(b)

All projection matrices except **I** are singular.

(c)

Eigenvalues of a diagonal matrix are its diagonal entries.

(d)

Positive determinant does not imply it's eigenvalues are all positive.

6.

$$z = 4x^2 + 12xy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 6 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Assume the eigenvalues of $\begin{bmatrix} 4 & 6 \\ 6 & c \end{bmatrix}$ are λ_1 and λ_2

$\Rightarrow$ If $0 < \lambda_1 < \lambda_2$, the graph of z is a bowl

$\Rightarrow \det(\begin{bmatrix} 4 & 6 \\ 6 & c \end{bmatrix}) = 4c - 36 = \lambda_1 \lambda_2 > 0$

$\Rightarrow c > 9$

$\Rightarrow$ And if $\lambda_1 < 0 < \lambda_2$, the graph of z has a saddle point

$\Rightarrow \det(\begin{bmatrix} 4 & 6 \\ 6 & c \end{bmatrix}) = 4c - 36 = \lambda_1 \lambda_2 < 0$

$\Rightarrow c < 9$

When $c = 9$, the graph of $z = (2x + 3y)^2$ is a "trough"
staying at zero along the line $2x + 3y = 0$