HW 2 solution

1.

$\mathbf{AA}^{-1} = \mathbf{I}$

$$=> \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First we focus on the first row of $\mathbf{A}$ and $\mathbf{I}$. By multiplication of matrices, it's easy to see that

$$=> \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now focus on the second row of $\mathbf{A}$ and $\mathbf{I}$. By multiplication of matrices, it's easy to see that

$$=> \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first row and the second row of $\mathbf{A}^{-1}$ can be found in the same way.

$$=> \mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } \mathbf{B} = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 7 & 0 & 0 \\ \hline 0 & 0 & 5 & 3 \\ 0 & 0 & 8 & 5 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D} \end{array}\right]$$

$$\mathbf{C}^{-1} = \frac{1}{\det(\mathbf{C})} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}, \quad \mathbf{D}^{-1} = \frac{1}{\det(\mathbf{D})} \begin{bmatrix} 5 & -3 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -8 & 5 \end{bmatrix}$$

$$\mathbf{B}^{-1} = \left[\begin{array}{c|c} \mathbf{C}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}^{-1} \end{array}\right] = \begin{bmatrix} 7 & -2 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -8 & 5 \end{bmatrix}$$

2.

Use Gauss-Jordan method to solve this problem

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 9 & 2 \\ 0 & 2 & 1 & 2 & 6 & 1 \\ 4 & 1 & 1 & -2 & 3 & 5 \end{array}\right] \xrightarrow[R_2(\frac{1}{2})]{R_{13}(-4)} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 9 & 2 \\ 0 & 1 & \frac{1}{2} & 1 & 3 & \frac{1}{2} \\ 0 & -11 & -3 & -6 & -23 & -3 \end{array}\right] \xrightarrow[R_{23}(11)]{R_{21}(-3)} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -2 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 & 3 & \frac{1}{2} \\ 0 & 0 & \frac{5}{2} & 5 & 10 & \frac{5}{2} \end{array}\right]$$

$$\xrightarrow{R_3(\frac{2}{5})} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -2 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 & 3 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right] \xrightarrow[R_{32}(-\frac{1}{2})]{R_{31}(\frac{1}{2})} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right]$$

$\therefore$ The solution is X= $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

3.

Assume the equality is true. Multiply both sides by A.

$$\mathbf{(A - B)^{-1} A = I + A^{-1}(B^{-1} - A^{-1})^{-1}}$$

Now we prove $\mathbf{(A - B)^{-1} A - A^{-1}(B^{-1} - A^{-1})^{-1} = I}$ is true.

$$\mathbf{(A - B)^{-1} A - A^{-1}(B^{-1} - A^{-1})^{-1}}$$

$$\mathbf{= [B(B^{-1} - A^{-1})A]^{-1} A - A^{-1}(B^{-1} - A^{-1})^{-1}}$$

$$\mathbf{= A^{-1}(B^{-1} - A^{-1})^{-1} B^{-1} A - A^{-1}(B^{-1} - A^{-1})^{-1}}$$

$$\mathbf{= A^{-1}(B^{-1} - A^{-1})^{-1}[B^{-1} A - I]}$$

$$\mathbf{= A^{-1}(B^{-1} - A^{-1})^{-1}[B^{-1} - A^{-1}]A}$$

$$\mathbf{= I}$$

Q.E.D.

4.

According to Gauss-Jordan method，the question can be rewritten as

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} -6 & f \\ d & -6 \\ e & -4 \end{bmatrix} = \begin{bmatrix} a & b \\ 3 & 0 \\ -4 & c \end{bmatrix}$$

$$\Rightarrow \begin{cases} -6+3d+4e=a \\ -12+d+5e=3 \\ 4d+2e=-4 \\ f-34=b \\ 2f-26=0 \\ -32=c \end{cases}$$

$$\therefore \text{ We can get } \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ -21 \\ -32 \\ -\frac{25}{9} \\ \frac{32}{9} \\ 13 \end{bmatrix}$$

5.

$$\mathbf{B = (I - A)(I + A)^{-1}}$$

$$\mathbf{=> B(I + A) = I - A}$$

$$\mathbf{=> B(I + A) + A = I}$$

$$\mathbf{=> B(I + A) + (I + A) = 2I}$$

$$\mathbf{=> (I + A)(I + B) = 2I}$$

$$\mathbf{=> (I + B) = (I + A)^{-1}\ 2I}$$

$$\mathbf{=> (I + B)^{-1} = \frac{I}{2}(I + A) =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

6.

$(I+BA^{-1})$ is invertible $\Leftrightarrow$ There exsists a matrix X such that $(I+BA^{-1})X=I$

$\Rightarrow (I+BA^{-1})X=I$

$\Rightarrow (AA^{-1}+BA^{-1})X=I$

$\Rightarrow (A+B)A^{-1}X=I$

$\Rightarrow X=A(A+B)^{-1}$.................$\because A$ and $(A+B)$ are invertible

$\Rightarrow$ We can find a matrix $X=A(A+B)^{-1}$ such that $(I+BA^{-1})X=I$

$\therefore$ $(I+BA^{-1})$ is invertible Q.E.D

To show $A(A+B)^{-1}B=B(A+B)^{-1}A=(A^{-1}+B^{-1})^{-1}$, we find the inverse of $A^{-1}+B^{-1}$ first.

$A^{-1}+B^{-1}=B^{-1}BA^{-1}+B^{-1}=B^{-1}(BA^{-1}+I)=B^{-1}(BA^{-1}+AA^{-1})=B^{-1}(B+A)A^{-1}$

or $A^{-1}+B^{-1}=A^{-1}+A^{-1}AB^{-1}=A^{-1}(I+AB^{-1})=A^{-1}(BB^{-1}+AB^{-1})=A^{-1}(B+A)B^{-1}$

$\Rightarrow A^{-1}+B^{-1}=B^{-1}(B+A)A^{-1}=A^{-1}(B+A)B^{-1}$

$\therefore A(A+B)^{-1}B=B(A+B)^{-1}A=(A^{-1}+B^{-1})^{-1}$ Q.E.D