HW 3 solution

1.

(a)

Assume the inverse of I-cA is I-dA

=>(I-cA)(I-dA)=I

=>$I-dA-cA+cdA^2=I-(c+d-cd)A=I$

=>c+d-cd=0

=>d=c/(c-1)

=>$(I-cA)^{-1}=I-\frac{c}{c-1}A$, $c\neq 1$

(b)

$$\mathbf{E}=\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}_{n\times n},\ \mathbf{E^2}=\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}=\begin{bmatrix}n & \cdots & n\\ \vdots & \ddots & \vdots\\ n & \cdots & n\end{bmatrix}=\mathbf{nE}$$

Assume the inverse of I-E is I-dE

=>(I-E)(I-dE)=I

=>$I-dE-E+dE^2=I-(d+1-nd)E=I$

=>d+1-nd=0

=>d=1/(n-1)

=>$(I-E)^{-1}=I-\frac{1}{n-1}E$, $n\neq 1$

2.

Given $B^T = -B$ and $A = (I+B)(I-B)^{-1}$

$$
\begin{aligned}
A^T &= \left[(I+B)(I-B)^{-1}\right]^T \\
&= \left[(I-B)^T\right]^{-1}(I+B)^T \\
&= (I^T - B^T)^{-1}(I^T + B^T) \\
&= (I+B)^{-1}(I-B) \\
&= \left[(I-B)^{-1}(I+B)\right]^{-1}
\end{aligned}
$$

$$
\begin{bmatrix}
\text{Proof}: (I-B)^{-1}(I+B) = (I+B)(I-B)^{-1} \\
(I-B)^{-1}(I+B) = (I-B)^{-1} + (I-B)^{-1}B \text{ and } (I+B)(I-B)^{-1} = (I-B)^{-1} + B(I-B)^{-1} \\
\Rightarrow (I-B)^{-1}B = \left[B^{-1}(I-B)\right]^{-1} = \left[B^{-1} - I\right]^{-1} = \left[(I-B)B^{-1}\right]^{-1} = B(I-B)^{-1}
\end{bmatrix}
$$

$\therefore A^T = \left[(I-B)^{-1}(I+B)\right]^{-1} = \left[(I+B)(I-B)^{-1}\right]^{-1} = A^{-1}$ Q.E.D.

3.

The solution is based on

http://ccjou.twbbs.org/blog/?p=6507 (每週問題 March 8, 2010)

$$
\left[\begin{array}{cccc|c}
\mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{17} & \mathbf{1} \\
\mathbf{3} & \mathbf{6} & \mathbf{-12} & \mathbf{3} & \mathbf{2} \\
\mathbf{2} & \mathbf{3} & \mathbf{-3} & \mathbf{2} & \mathbf{3} \\
\mathbf{0} & \mathbf{2} & \mathbf{-2} & \mathbf{6} & \mathbf{4}
\end{array}\right]
\sim
\left[\begin{array}{cccc|c}
\mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{17} & \mathbf{1} \\
\underline{\mathbf{3}} & \mathbf{0} & \mathbf{-24} & \mathbf{-48} & \mathbf{2} \\
\underline{\mathbf{2}} & \mathbf{-1} & \mathbf{-11} & \mathbf{-32} & \mathbf{3} \\
\underline{\mathbf{0}} & \mathbf{2} & \mathbf{-2} & \mathbf{6} & \mathbf{4}
\end{array}\right]
\sim
\left[\begin{array}{cccc|c}
\mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{17} & \mathbf{1} \\
\underline{\mathbf{2}} & \mathbf{-1} & \mathbf{-11} & \mathbf{-32} & \mathbf{3} \\
\underline{\mathbf{3}} & \mathbf{0} & \mathbf{-24} & \mathbf{-48} & \mathbf{2} \\
\underline{\mathbf{0}} & \mathbf{2} & \mathbf{-2} & \mathbf{6} & \mathbf{4}
\end{array}\right]
\sim
$$

$$
\left[\begin{array}{cccc|c}
\mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{17} & \mathbf{1} \\
\underline{\mathbf{2}} & \mathbf{-1} & \mathbf{-11} & \mathbf{-32} & \mathbf{3} \\
\underline{\mathbf{3}} & \underline{\mathbf{0}} & \mathbf{-24} & \mathbf{-48} & \mathbf{2} \\
\underline{\mathbf{0}} & \mathbf{-}\underline{\mathbf{2}} & \mathbf{-24} & \mathbf{-58} & \mathbf{4}
\end{array}\right]
\sim
\left[\begin{array}{cccc|c}
\mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{17} & \mathbf{1} \\
\underline{\mathbf{2}} & \mathbf{-1} & \mathbf{-11} & \mathbf{-32} & \mathbf{3} \\
\underline{\mathbf{3}} & \underline{\mathbf{0}} & \mathbf{-24} & \mathbf{-48} & \mathbf{2} \\
\underline{\mathbf{0}} & \mathbf{-}\underline{\mathbf{2}} & \underline{\mathbf{1}} & \mathbf{-10} & \mathbf{4}
\end{array}\right]
$$

$$
\therefore \mathbf{P} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix},
\mathbf{L} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} & \mathbf{1} & \mathbf{1} \end{bmatrix},
\mathbf{U} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{17} \\ \mathbf{0} & \mathbf{-1} & \mathbf{-11} & \mathbf{-32} \\ \mathbf{0} & \mathbf{0} & \mathbf{-24} & \mathbf{-48} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-10} \end{bmatrix}
$$

4.

$A = LU$ cne be obtained by Gaussian elimination and each step of row operation can be written as elimination matrix E

Therefore , $L = E_1 \times E_2 \cdots E_n$ (We do n times row operation)

$\Rightarrow L^{-1} = E_n^{-1} \times E_{n-1}^{-1} \cdots E_1^{-1}$

E^{-1} is adding negative sign to the non-diagonal entries of E .

Therefore , it still has integral entries.

So , L^{-1} is still an integral matrix.

On the other hand , we can write $A^T = U^T L^T$.

U^T is each step of row operation on A^T , it can be represented as $E_1^T \times E_2^T \cdots E_n^T$.

For the same reason , $(U^T)^{-1}$ is an integral matrix , too.

Therefore , U^{-1} is also an integral matrix.

$A^{-1} = U^{-1} L^{-1}$

Both U^{-1} and L^{-1} are integral matrices , so we can conclude that A^{-1} is also an integral matrix.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -9 & 1 & 0 \\ 100 & 57 & 1 \end{bmatrix} \begin{bmatrix} 1 & -73 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -73 & 4 \\ -9 & 658 & -28 \\ 100 & -7243 & 857 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 361102 & 33589 & -588 \\ 4913 & 457 & -8 \\ -613 & -57 & 1 \end{bmatrix}$$

5.

A is symmetric and possesses an LDU factorization

=> A=LDU

=>$A^T=(LDU)^T=U^TD^TL^T=U^TDL^T$ (The off-diagonal elements of D are zeros. $\therefore D^T=D$)

A is symmetric

=> $A^T=A$

=> $U^TDL^T=LDU$

=>$L^T=U$

=>$A=LDL^T$

6.

Use Gauss-Jordan method to find the inverce of $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cc|cc} A & 0 & I_m & 0 \\ B & C & 0 & I_n \end{array}\right] \xrightarrow{R_1(A^{-1})} \left[\begin{array}{cc|cc} I_m & 0 & A^{-1} & 0 \\ B & C & 0 & I_n \end{array}\right] \xrightarrow{R_{12}(-B)} \left[\begin{array}{cc|cc} I_m & 0 & A^{-1} & 0 \\ 0 & C & -BA^{-1} & I_n \end{array}\right]$$

$$\xrightarrow{R_2(C^{-1})} \left[\begin{array}{cc|cc} I_m & 0 & A^{-1} & 0 \\ 0 & I_n & -C^{-1}BA^{-1} & C^{-1} \end{array}\right]$$

$\therefore$ If A and C are invertible , the matrix $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ must be invertible and its inverce is $\begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$