Solutions to Homework 4

1.

Ans: (a)(d)(e)

(a) A plane passing through the origin.

(b) A plane which does not pass through the origin.

ex.

Let $S = \{(b_1, b_2, b_3) \mid b_1 = 1\}$

$v_1 = (1,2,1) \in S$, but $2v_1 = (2,4,2) \notin S$

(c) Three planes formed by three axes.

ex.

Let $S = \{(b_1, b_2, b_3) \mid b_1 b_2 b_3 = 0\}$

$v_1 = (1,0,1) \in S$,

$v_2 = (0,1,0) \in S$

but $v_1 + v_2 = (1,1,1) \notin S$

(d) A plane passing through the origin.

(e) A plane passing through the origin.

(f) ex.

Let $S = \{(b_1, b_2, b_3) \mid b_1 \le b_2 \le b_3\}$

$v_1 = (1,2,3) \in S$,

but $-v_1 = (-1,-2,-3) \notin S$

2.

(a)False

$\because$ The vectors $\mathbf{b}$ don't contain the zero vector.

(b)True

Only the zero matrix has C(A)=$\{0\}$.

(c)True

$C(2A) = C(A)$.

(d)False

If A=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then C(A)=$R^2$.

$\Rightarrow A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow C(A-I) = \{0\}$

$\Rightarrow C(A) \ne C(A-I)$

3.

(a)

If $\mathbf{u}$ and $\mathbf{v}$ are both in $\mathbf{S}+\mathbf{T}$, then $\mathbf{u}=\mathbf{s}_1+\mathbf{t}_1$ and $\mathbf{v}=\mathbf{s}_2+\mathbf{t}_2$

Addition:

$\mathbf{u}+\mathbf{v}=(\mathbf{s}_1+\mathbf{t}_1)+(\mathbf{s}_2+\mathbf{t}_2)=(\mathbf{s}_1+\mathbf{s}_2)+(\mathbf{t}_1+\mathbf{t}_2)$

$\because (\mathbf{s}_1+\mathbf{s}_2)\in\mathbf{S}$ and $(\mathbf{t}_1+\mathbf{t}_2)\in\mathbf{T}$

$\therefore \mathbf{u}+\mathbf{v}\in\mathbf{S}+\mathbf{T}$

Scaling:

Let c be a scaler, $c\in R$

$c\mathbf{u}=c(\mathbf{s}_1+\mathbf{t}_1)=c\mathbf{s}_1+c\mathbf{t}_1$

$\because c\mathbf{s}_1\in\mathbf{S}$ and $c\mathbf{t}_1\in\mathbf{T}$

$\therefore c\mathbf{u}\in\mathbf{S}+\mathbf{T}$

(b)

If $\mathbf{S}$ and $\mathbf{T}$ are different lines, then $\mathbf{S}\cup\mathbf{T}$ is just the two lines but $\mathbf{S}+\mathbf{T}$ is the whole plane that they span.

4.

(a)

We know that the column space of a matrix is formed by the columns , and the columns of A and B are also the columns of X.

$\therefore C(X)=C(A)\cup C(B)$

(b)

$\forall$ vector $\mathbf{u}\in N(X)$

$\Rightarrow X\mathbf{u}=0$

$\Rightarrow \begin{bmatrix} A \\ B \end{bmatrix}\mathbf{u}=0$

$\Rightarrow \begin{bmatrix} A\mathbf{u} \\ B\mathbf{u} \end{bmatrix}=0$

$\Rightarrow \mathbf{u}\in N(A)\cap N(B)$

$\Rightarrow N(X)\subset N(A)\cap N(B)$

$\forall$ vector $\mathbf{v}\in N(A)\cap N(B)$

$\Rightarrow \begin{bmatrix} A\mathbf{v} \\ B\mathbf{v} \end{bmatrix}=0$

$\Rightarrow \begin{bmatrix} A \\ B \end{bmatrix}\mathbf{v}=0$

$\Rightarrow X\mathbf{v}=0$

$\Rightarrow$ vector $\mathbf{v}\in N(X)$

$\Rightarrow N(A)\cap N(B)\subset N(X)$

$\because N(A)\cap N(B)\subset N(X)$ and $N(X)\subset N(A)\cap N(B)$

$\therefore N(X)=N(A)\cap N(B)$

5.

$$\mathbf{RX} = \left[\begin{array}{c|cccccc} 1 & -3 & 0 & 5 & 0 & 0 & 7 \\ 0 & 0 & 1 & -4 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

=>rank of R=4

=>(7-4) free variables (x_2, x_4, x_7)

$$\Rightarrow \mathbf{N} = \begin{bmatrix} 3 & -5 & -7 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 4 & -3 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & -2 \\ 0 & 0 & -1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Note that the free variables are in bold face.

6.

(a)

$\because R$ is m by n of rank r

$$\Rightarrow \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } (n-r) \\ (m-r) \text{ by } r & (m-r) \text{ by } (n-r) \end{bmatrix}$$

(b)Find the right-inverse B with $RB = I$ if $r = m$.

$rank(R) = m$

$\Rightarrow R = \begin{bmatrix} I & F \end{bmatrix}$

$\Rightarrow$ We can choose a matrix B to let the linear combination of the columns in R form an identity matrix I.

$\Rightarrow$ So it is obvious to show that $B = \begin{bmatrix} I \\ 0 \end{bmatrix}$ where I is m by m and B is n by m .

(c)Find a left-inverse C with $CR = I$ if $r = n$

$rank(R) = n$

$\Rightarrow R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $R^T = \begin{bmatrix} I & 0 \end{bmatrix}$

$\Rightarrow R^T C^T = I$

$\Rightarrow$ We can also choose a matrix C^T to let the linear combination of the columns in R^T form an identity matrix I.

$\Rightarrow$ So it is obvious to show that $C^T = \begin{bmatrix} I \\ D^T \end{bmatrix}$ where I is n by n , D^T is an any $(m - n)$ by n matrix .

$\Rightarrow C = \begin{bmatrix} I & D \end{bmatrix}$, C is n by m .

(d)

$$R^T = \begin{bmatrix} I & 0 \\ F^T & 0 \end{bmatrix}$$

$\Rightarrow R^T$ is n by m , I is r by r and F^T is $(n - r)$ by r.

$$\Rightarrow \begin{bmatrix} I & 0 \\ F^T & 0 \end{bmatrix} \xrightarrow{R_{12}(-F^T)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ The reduced row echelon form of $R^T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

(e)

$$R^T R = \begin{bmatrix} I & F \\ F^T & F^T F \end{bmatrix}$$

$\Rightarrow R^T R$ is n by n , I is r by r , F^T is $(n\text{-}r)$ by r and $F^T F$ is $(n\text{-}r)$ by $(n\text{-}r)$.

$$\Rightarrow \begin{bmatrix} I & F \\ F^T & F^T F \end{bmatrix} \xrightarrow{R_{12}(-F^T)} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ The reduced row echelon form of $R^T R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(f)

$\because$ The reduced row echelon form of $R^T R$ is the same as R

$\Rightarrow$ The nullspace matrix N of $R^T R$ is the same as R

$$\Rightarrow N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$\therefore N(R^T R) = N(R)$

Q.E.D.