Solutions to Homework 5

1.

(a)

N(A)= $\begin{bmatrix} 2 & 3 & 1 & 0 \end{bmatrix}^T$

rank(A) = n-dim N(A)= 4-1=3

The complete solution is $\mathbf{x} = t\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, t \in R.$

(b)

Recall the method of finding N(A) from the reduced echelon form of A.

(If A ~ $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$, then N(A) = $\begin{bmatrix} -F \\ I \end{bmatrix}$)

We can get A ~ $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c)

A is full colum rank => C(A) = R^3

$\therefore$ Any **b** can be a linear combination of columns of A.

$\therefore$ A**x = b** is solvable for all **b**.

2.

(a)

A subspace in $\mathbf{R^4} = \left\{ \mathbf{x} = \begin{bmatrix} x_1 & x_1 & x_1 & x_1 \end{bmatrix}^T \mid \mathbf{x} \in \mathbf{R^4} \right\}$

$\Rightarrow$ Any vectors in this subspace are $\begin{bmatrix} x_1 & x_1 & x_1 & x_1 \end{bmatrix}^T = x_1 \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$

$\Rightarrow x_1$ is any constant , so the vectors in this subspace are the linear combinations of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$

$\therefore$ The basis is $\left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \right\}$

(b)

A subspace in $\mathbf{R^4} = \left\{ \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \mid x_1 + x_2 + x_3 + x_4 = 0 \ , \ \mathbf{x} \in \mathbf{R^4} \right\}$

$\Rightarrow$ Let $x_2 = C_1$, $x_3 = C_2$, $x_4 = C_3$ and C_1 , C_2 , $C_3 \in \Re$

$\Rightarrow x_1 = -C_1 - C_2 - C_3$

$$\Rightarrow \mathbf{x} = C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \mathbf{x}$ is the linear combination of $\left\{ \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T \ , \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^T \ , \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}^T \right\}$

$\therefore$ The basis is $\left\{ \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T \ , \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^T \ , \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}^T \right\}$

(c)

Let $\mathbf{u_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{u_2} = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T$

A subspace in $\mathbf{R^4} = \left\{ \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \mid \mathbf{x}^T \mathbf{u_1} = \mathbf{x}^T \mathbf{u}_2 = 0 \ , \ \mathbf{x} \in \mathbf{R^4} \right\}$

$\Rightarrow$ We can rewrite the relation between $\mathbf{x}$, $\mathbf{u_1}$ and $\mathbf{u_1}$ as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

$\Rightarrow$ This subspace is equal to the left null space of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ The left nullspace matrix = $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ The basis is $\left\{ \begin{bmatrix} 1 & -1 & -1 & 0 \end{bmatrix}^T \ , \begin{bmatrix} 0 & 0 & -1 & - \end{bmatrix}^T \right\}$

(d)

A subspace in $\mathbf{R}^4 = \left\{ \mathbf{x} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \mid c = a+b \text{ and } d = c-b \text{ , } \mathbf{x} \in \mathbf{R}^4 \right\}$

$\Rightarrow \mathbf{x} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T = \begin{bmatrix} a & b & a+b & a \end{bmatrix}^T = a \times \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} + b \times \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$

$\therefore$ The basis is $\left\{ \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T \text{ , } \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \right\}$

3.

$$I = \underset{P_1}{\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}} - \underset{P_2}{\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}} + \underset{P_3}{\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}} + \underset{P_4}{\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}} - \underset{P_5}{\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}$$

To show the independence of these P's, assume $c_1P_1 + \dots c_5P_5 = 0$

Focus on P_1, $p_{33} = 1$ but other p_{33}'s are zero, so c_1 must be zero.

Focus on P_2, $p_{12} = 1$ but other p_{12}'s are zero(except P_1, but c_1=0 already), so c_2 must be zero.

Focus on P_3, $p_{22} = 1$ but other p_{22}'s are zero, so c_3 must be zero.

Focus on P_4, $p_{11} = 1$ but other p_{11}'s are zero, so c_4 must be zero.

Focus on P_5, $p_{32} = 1$ but other p_{32}'s are zero(except P_4, but c_4=0 already), so c_5 must be zero.

$\because c_1 = \dots = c_5 = 0 \;\; \therefore$ P's are independent.

Let Q be a linear combination of these P's.

$\because$ There is an *ONE* in each row and each column of these P's.

$\therefore$ Summations of rows and columns of Q will be all equal.

$\therefore$ P's can be a basis for matrices with this property.

4.

(a)

$$Ans:\begin{bmatrix}1 & 0\\1 & 0\\0 & 2\end{bmatrix}$$

The column space is span $\left\{\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 0 & 2\end{bmatrix}^T\right\}$

The row space is span $\left\{\begin{bmatrix}1 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 2\end{bmatrix}^T\right\}$ which contains $\left\{\begin{bmatrix}1 & 2\end{bmatrix}^T, \begin{bmatrix}2 & 3\end{bmatrix}^T\right\}$

(b)

$$Ans:\begin{bmatrix}1 & 1 & -1\\1 & 1 & -1\\3 & 3 & -3\end{bmatrix}$$

The basis of column space is $\left\{\begin{bmatrix}1 & 1 & 3\end{bmatrix}^T\right\}$

The basis of null space is $\left\{\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T, \begin{bmatrix}0 & 1 & 1\end{bmatrix}^T\right\}$

(c)

Dimention of null space $=$ the number of columns $-$ rank

Dimention of left null space $=$ the number of rows $-$ rank

$\Rightarrow$ the number of rows $+1=$ the number of columns

$$Ans:\begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\1 & 3 & 1\\0 & 2 & 2\end{bmatrix}$$

(d)

$$\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\a & b\end{bmatrix}=0$$

$$\Rightarrow a=-1\,,\, b=-\frac{1}{2}$$

$$Ans:\begin{bmatrix}2 & 1\\-1 & -\frac{1}{2}\end{bmatrix}$$

(e)

Row space = column space $\Rightarrow$ The matrix must be square

$\Rightarrow$

Dimention of null space = the number of columns $-$ rank

Dimention of left null space = the number of rows $-$ rank

But the matrix is square and dimention of column space = dimention of row space.

$\therefore$ This matrix is impossible

5.

dim N(A) = 2 (There are two homogeneous solutions)

$\therefore$ A basis for the null space can be $\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\}$. ----(c)

A must be 4 by 3.

From the rank theorom, rank(A) = n-dim N(A)=1

Since dim C(A) = rank(A) =1, a basis for the column space of A contains only one vector.

$\because$ Ax = $[1\ 2\ 1\ 1]^T$ is sovable $\therefore [1\ 2\ 1\ 1]^T$ is in C(A)

$\therefore$ A basis for the column space can be $\begin{bmatrix}1\\2\\1\\1\end{bmatrix}$ ----(b)

We know N(A) = $\begin{bmatrix}2 & 1\\ \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$, free variables are in bold face.

Recall the method of finding N(A) from the reduced echelon form of A.

(If A ~ $\begin{bmatrix}I & F\\0 & 0\end{bmatrix}$, then N(A) = $\begin{bmatrix}-F\\I\end{bmatrix}$)

We can get A ~ $\begin{bmatrix}1 & -2 & -1\\0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$----(a)

6.

The reduced row echelon form of A is $\begin{bmatrix} 1 & 3 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ From the reduced row echelon form , we can get the row space and the null space of A.

$\therefore \text{span}\left\{ \begin{bmatrix} 1 & 3 & 0 & -2 & -1 \end{bmatrix}^T \ , \ \begin{bmatrix} 0 & 0 & 1 & 1 & 3 \end{bmatrix}^T \right\}$ is the row space

$\Rightarrow$ the nullspace matrix is $\begin{bmatrix} -3 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \text{span}\left\{ \begin{bmatrix} -3 & 1 & 0 & 0 & 0 \end{bmatrix}^T \ , \ \begin{bmatrix} 2 & 0 & -1 & 1 & 0 \end{bmatrix}^T \ , \ \begin{bmatrix} 1 & 0 & -3 & 0 & 1 \end{bmatrix}^T \right\}$ is the null space

To find the column space and the left column space , we recover A first .

$A = E \times (\text{row echelon form of } A)$ and $I_3 = E \times \begin{bmatrix} 1 & -2 & -3 \\ 2 & -3 & -2 \\ 3 & -6 & -8 \end{bmatrix}$

$$\Rightarrow E = \begin{bmatrix} 1 & -2 & -3 \\ 2 & -3 & -2 \\ 3 & -6 & -8 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & 2 & -5 \\ 10 & 1 & -4 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\therefore A = E \times (\text{row echelon form of } A) = \begin{bmatrix} 12 & 36 & 2 & -22 & -6 \\ 10 & 30 & 1 & -19 & -7 \\ -3 & -9 & 0 & 6 & 3 \end{bmatrix}$$

$\because$ The row operation doesn't change the relation between the columns of A

$\Rightarrow$

$\text{column}_2 = \text{column}_1 \times 3$

$\text{column}_4 = \text{column}_1 \times (-2) + \text{column}_3$

$\text{column}_5 = \text{column}_1 \times (-1) + \text{column}_3 \times 3$

$\therefore \text{span}\left\{ \begin{bmatrix} 12 & 10 & -3 \end{bmatrix}^T \ , \ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T \right\}$ is the column space of A

$\because E^{-1} \times A =$ (row echelon form of A)

$$\Rightarrow \begin{bmatrix} 1 & -2 & -3 \\ 2 & -3 & -2 \\ 3 & -6 & -8 \end{bmatrix} \times \begin{bmatrix} 12 & 36 & 2 & -22 & -6 \\ 10 & 30 & 1 & -19 & -7 \\ -3 & -9 & 0 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We focus on the zero row of the row echelon form of A

$$\Rightarrow \begin{bmatrix} 3 & -6 & -8 \end{bmatrix} \times \begin{bmatrix} 12 & 36 & 2 & -22 & -6 \\ 10 & 30 & 1 & -19 & -7 \\ -3 & -9 & 0 & 6 & 3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3 & -6 & -8 \end{bmatrix}^T \in \text{the left null space of } A$$

We know that the dimention of left null space $=$ the number of rows $-$ rank and rank(A) $= 2$

$\Rightarrow$ the dimention of left null space $= 1$

$\therefore \text{span}\left\{ \begin{bmatrix} 3 & -6 & -8 \end{bmatrix}^T \right\}$ is the left null space