# Solutions to Homework 6

1.

(a)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Ax}$$

$$R(T) = C(\mathbf{A}) = span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$$

$$\ker(T) = N(\mathbf{A}) = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$$

$(b)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{Ax}$$

$$R(T) = C(\mathbf{A}) = span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$

$$\ker(T) = N(\mathbf{A}) = span\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\}$$

$(c)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Ax}$$

$$R(T) = C(\mathbf{A}) = span\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$$

$$\ker(T) = N(\mathbf{A}) = span\{0\}$$

$(d)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{Ax}$$

$$R(T) = C(\mathbf{A}) = span\{0\}$$

$$\ker(T) = N(\mathbf{A}) = R^3$$

2.

Assume $\mathbf{A}$ is the matrix represented T

$\Rightarrow$ If $\mathbf{x} \in \mathbf{R}^n$ , $\mathbf{y} \in \mathbf{R}^m$

$\Rightarrow$ $\mathbf{Ax} = \mathbf{y}$ , $\mathbf{A}$ is m by n

(a)If T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$

$\Rightarrow$ All vectors in $\mathbf{R}^m$ will be mapped

$\Rightarrow$ $Range(T) = C(\mathbf{A}) = \mathbf{R}^m$

$\Rightarrow$ $rank(\mathbf{A}) = m$ , $\mathbf{A}$ is full row rank

$\Rightarrow$ n $\geq$ m

(b)If T is one-to-one

$\Rightarrow$ Each vector in $\mathbf{R}^m$ will be mapped to only one vector in $\mathbf{R}^n$

$\Rightarrow$ The null space of $\mathbf{A}$ must be the zero vector , because only one vector can be mapped to zero

$\Rightarrow$ $\mathbf{A}$ is full column rank

$\Rightarrow$ m $\geq$ n

(c)If T is one-to-one and maps $\mathbf{R}^n$ onto $\mathbf{R}^m$

$\Rightarrow$ According to (a) and (b) , the matrix $\mathbf{A}$ must satisfy n $\geq$ m and m $\geq$ n

$\therefore$ If m = n , T is one-to-one and maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ .

3.

Linear transformation keep straight lines straight. Two parallel edges of a square (edges differing by a fixed $v$) go to two parallel edges (edges differing by $T(v)$). So the output is a parallelogram.

ex.

Given $T$: $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & -1 \end{bmatrix}$ and two vectors in $R^2$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Note that $\| v_1 \| = \| v_2 \|$ and $\| v_1 \| // \| v_2 \|$

$$v_1' = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & -1 \end{bmatrix} v_1 = \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}$$

$$v_2' = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & -1 \end{bmatrix} v_2 = \begin{bmatrix} -5 \\ -7 \\ 0 \end{bmatrix}$$

=> $\| v_1' \| = \| v_2' \|$ and $\| v_1' \| // \| v_2' \|$

After the linear transformation, two parallel vectors with the same length are still parallel and have the same length.

4.

The coordinate vector of $1-t+2t^2=\begin{bmatrix}1 & -1 & 2\end{bmatrix}^t$

The coordinate vector of $2-t+5t^2=\begin{bmatrix}2 & -1 & 5\end{bmatrix}^t$

The coordinate vector of $-1+4t+2t^2=\begin{bmatrix}-1 & 4 & 2\end{bmatrix}^t$

Let a matrix be $\begin{bmatrix}1 & 2 & -1\\ -1 & -1 & 4\\ 2 & 5 & 2\end{bmatrix}$ and the reduced row echelon form is $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$\Rightarrow$ The columns are independent

$\therefore 1-t+2t^2$ , $2-t+5t^2$ and $-1+4t+2t^2$ are independent in $\mathbf{P}^2$ .

5.

$(a)$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\\0\end{bmatrix} \quad => \begin{bmatrix}1 & * & *\\ 1 & * & *\\ 0 & * & *\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\3\\2\end{bmatrix} \quad => \begin{bmatrix}1 & 2 & *\\ 1 & 3 & *\\ 0 & 2 & *\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\3\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad => \begin{bmatrix}1 & 2 & -2\\ 1 & 3 & -4\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

The matrix representation of $T$ is $\begin{bmatrix}1 & 2 & -2\\ 1 & 3 & -4\\ 0 & 2 & -3\end{bmatrix}$.

$(b)$

Let $\mathbf{B}$ be the basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

$$\mathbf{A}' : [\mathbf{v}]_\mathbf{B} \xrightarrow{\mathbf{B}} \mathbf{v} \xrightarrow{\mathbf{A}} \mathbf{T(v)} \xrightarrow{\mathbf{B^{-1}}} [\mathbf{T(v)}]_\mathbf{B}$$

$$\therefore \mathbf{A}' = \mathbf{B^{-1}AB} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 2 & -2\\ 1 & 3 & -4\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 2\\ 1 & 1 & 1\\ 0 & 2 & -1\end{bmatrix}$$

(*c*)

Let **B** be the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

$$\mathbf{A}^{'} : [\mathbf{v}]_{\mathbf{B}} \xrightarrow{\mathbf{B}} \mathbf{v} \xrightarrow{\mathbf{A}} \mathbf{T(v)} \xrightarrow{\mathbf{B^{-1}}} [\mathbf{T(v)}]_{\mathbf{B}}$$

$$\therefore \mathbf{A}^{'} = \mathbf{B^{-1}AB} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

6.

Let a basis $\mathbf{V}=\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and another basis $\mathbf{W}=\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

Assume T is a linear transformation about mapping the coordinates of **V** to the coordinates of **W** .

$\Rightarrow \mathbf{A}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ where **A** is the matrix represented T .

$$\therefore \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

If $\mathbf{Av} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{v}_1 - \mathbf{v}_2$$