# Solutions to Homework 7

1.

(a)

$$B=\{b_1,b_2\}=\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\},T(b_1)=\begin{bmatrix}3\\2\\3\end{bmatrix},T(b_2)=\begin{bmatrix}2\\0\\2\end{bmatrix}$$

$$\because [\mathbf{x}]_B=\begin{bmatrix}3\\1\end{bmatrix}$$

$$\therefore \mathbf{x}=3b_1+b_2$$

$$\Rightarrow T(\mathbf{x})=3T(b_1)+T(b_2)=\begin{bmatrix}11\\6\\11\end{bmatrix}$$

(*b*)

$$C=\{\mathbf{c_1},\mathbf{c_2}\}=\{\begin{bmatrix}5\\4\\5\end{bmatrix},\begin{bmatrix}-5\\-6\\-5\end{bmatrix}\}$$

$$b_1=\begin{bmatrix}1\\1\\1\end{bmatrix}=0.1\begin{bmatrix}5\\4\\5\end{bmatrix}-0.1\begin{bmatrix}-5\\-6\\-5\end{bmatrix}=0.1\mathbf{c_1}-0.1\mathbf{c_2}$$

$$b_2=\begin{bmatrix}1\\0\\1\end{bmatrix}=0.6\begin{bmatrix}5\\4\\5\end{bmatrix}+0.4\begin{bmatrix}-5\\-6\\-5\end{bmatrix}=0.6\mathbf{c_1}+0.4\mathbf{c_2}$$

$$\therefore [B]_C=\{\begin{bmatrix}0.1\\-0.1\end{bmatrix},\begin{bmatrix}0.6\\0.4\end{bmatrix}\}$$

$$\Rightarrow [\mathbf{x}]_C=[B]_C[\mathbf{x}]_B=\begin{bmatrix}0.1&0.6\\-0.1&0.4\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}0.9\\0.1\end{bmatrix}$$

$$T(\mathbf{x})=\begin{bmatrix}11\\6\\11\end{bmatrix}=3.6\begin{bmatrix}5\\4\\5\end{bmatrix}+1.4\begin{bmatrix}-5\\-6\\-5\end{bmatrix}=3.6\mathbf{c_1}+1.4\mathbf{c}_2$$

$$\Rightarrow [T(\mathbf{x})]_C=\begin{bmatrix}3.6\\1.4\end{bmatrix}$$

(*c*)

$$[B]_C=\{\begin{bmatrix}0.1\\-0.1\end{bmatrix},\begin{bmatrix}0.6\\0.4\end{bmatrix}\}$$

$(d)$

Choose $[\mathbf{x}_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [\mathbf{x}_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore T(\mathbf{x}_1) = T(\mathbf{b}_1) = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$

$\therefore T(\mathbf{x}_2) = T(\mathbf{b}_2) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}_B$

Let **A** be matrix representation for T with respect to basis $B$

$\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow \mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

$(e)$

$C=\{\mathbf{c_1},\mathbf{c_2}\}=\{\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ -6 \\ -5 \end{bmatrix}\}=\{4\mathrm{b}_1+\mathrm{b}_2, -6\mathrm{b}_1+\mathrm{b}_2\}$

$T(\mathbf{c_1}) = 4T(\mathrm{b}_1)+T(\mathrm{b}_2) = \begin{bmatrix} 14 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 1.6 \end{bmatrix}_C$

$T(\mathbf{c}_2) = -6T(\mathrm{b}_1)+T(\mathrm{b}_2) = \begin{bmatrix} -16 \\ -12 \\ -16 \end{bmatrix} = \begin{bmatrix} -3.6 \\ -0.4 \end{bmatrix}_C$

Choose $[\mathbf{x}_1]_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [\mathbf{x}_2]_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore T(\mathbf{x}_1) = T(\mathbf{c}_1) = \begin{bmatrix} 14 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 1.6 \end{bmatrix}_C$

$\therefore T(\mathbf{x}_1) = T(\mathbf{c}_2) = \begin{bmatrix} -16 \\ -12 \\ -16 \end{bmatrix} = \begin{bmatrix} -3.6 \\ -0.4 \end{bmatrix}_C$

Let **A** be matrix representation for T with respect to basis $C$

$\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4.4 & -3.6 \\ 1.6 & -0.4 \end{bmatrix}$

$\Rightarrow \mathbf{A} = \begin{bmatrix} 4.4 & -3.6 \\ 1.6 & -0.4 \end{bmatrix}$

2.

($a$) Find the change-of-coordinates matrix from $B$ to $C$.

If there are three vectors in this subspace which are $\mathbf{x_1}$ , $\mathbf{x_2}$ and $\mathbf{x_3}$.

Let

$\mathbf{x_1} = \mathbf{b_1} = 4\mathbf{c_1} - \mathbf{c_2}$

$\mathbf{x}_2 = \mathbf{b_2} = -\mathbf{c_1} + \mathbf{c_2} + \mathbf{c_3}$

$\mathbf{x}_3 = \mathbf{b_3} = \mathbf{c_2} - 2\mathbf{c_3}$

such taht

$[\mathbf{x_1}]_B = [1 \quad 0 \quad 0]^t$ and $[\mathbf{x_1}]_C = [4 \quad -1 \quad 0]^t$

$[\mathbf{x}_2]_B = [0 \quad 1 \quad 0]^t$ and $[\mathbf{x_1}]_C = [-1 \quad 1 \quad 1]^t$

$[\mathbf{x}_3]_B = [0 \quad 0 \quad 1]^t$ and $[\mathbf{x}_3]_C = [0 \quad 1 \quad -2]^t$

$$\Rightarrow [\mathbf{x_1} \quad \mathbf{x}_2 \quad \mathbf{x}_3] = [\mathbf{b_1} \quad \mathbf{b_2} \quad \mathbf{b}_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{c_1} \quad \mathbf{c_2} \quad \mathbf{c}_3]\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

And $[B]_C \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$$\Rightarrow [B]_C = [\mathbf{c_1} \quad \mathbf{c_2} \quad \mathbf{c_3}]^{-1}[\mathbf{b_1} \quad \mathbf{b_2} \quad \mathbf{b_3}] = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$\therefore$ the $2^{\text{nd}}$ column of the change-of-coordinates matrix from $B$ to $C = [-1 \quad 1 \quad 1]^t$

($b$)

According to (a) , we know that $[\mathbf{b_1} \quad \mathbf{b}_2 \quad \mathbf{b}_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{c_1} \quad \mathbf{c}_2 \quad \mathbf{c}_3]\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

And $[C]_B \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow [C]_B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0.4 \\ 0.1 & 0.4 & -0.3 \end{bmatrix}$$

$\therefore$ the 3rd column of the change-of-coordinates matrix from $C$ to $B = \begin{bmatrix} 0.1 & 0.4 & -0.3 \end{bmatrix}^t$

3.

$(a)$

The point is the projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ onto **W**.

$$\Rightarrow \mathbf{W}(\mathbf{W}^\mathbf{T}\mathbf{W})^{-1}\mathbf{W}^\mathbf{T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8810 \\ 0.1905 \\ -0.2619 \end{bmatrix}$$

$(b)$

$$P_W = \mathbf{W}(\mathbf{W}^\mathbf{T}\mathbf{W})^{-1}\mathbf{W}^\mathbf{T} = \begin{bmatrix} 0.8810 & 0.1905 & -0.2619 \\ 0.1905 & 0.6952 & 0.4190 \\ -0.2619 & 0.4190 & 0.4238 \end{bmatrix}$$

rank $P_W = 2$

$(c)$

$$P_{W^\perp} = \mathbf{I} - P_W = \begin{bmatrix} 0.1190 & -0.1905 & 0.2619 \\ -0.1905 & 0.3048 & -0.4190 \\ 0.2619 & -0.4190 & 0.5762 \end{bmatrix}$$

rank $P_{W^\perp} = 1$

$(d)$

$\mathbf{W}^\perp = \mathrm{C}(P_{W^\perp})$

$\Rightarrow$ A basis for $P_{W^\perp}$ can be $\begin{bmatrix} 0.1190 \\ -0.1905 \\ 0.2619 \end{bmatrix}$

4.

$\mathbf{P}$ is an orthogonal projection matrix if $(\mathbf{I}-\mathbf{P})\mathbf{x} \perp \mathbf{Py}$

$\Rightarrow$

$$\left[(\mathbf{I}-\mathbf{P})\mathbf{x}\right]^t \mathbf{Py} = \mathbf{x}^t(\mathbf{I}-\mathbf{P}^t)\mathbf{Py}$$
$$= \mathbf{x}^t(\mathbf{I}-\mathbf{P})\mathbf{Py} \text{.............}(\because \mathbf{P}^t = \mathbf{P})$$
$$= \mathbf{x}^t\mathbf{Py} - \mathbf{x}^t\mathbf{P}^2\mathbf{y}$$
$$= \mathbf{x}^t\mathbf{Py} - \mathbf{x}^t\mathbf{Py} \text{............}(\because \mathbf{P}^2 = \mathbf{P})$$
$$= 0$$

$\therefore (\mathbf{I}-\mathbf{P})\mathbf{x} \perp \mathbf{Py}$ Q.E.D.

5.

$(a)$

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

The columns of $\mathbf{A}$ are all independent

$$\therefore \text{C}(\mathbf{A}) = \text{span}\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\} = \text{span}\{\mathbf{a}\}$$

$$\Rightarrow \text{P}_\text{C} = \frac{\mathbf{aa}^T}{\mathbf{a}^T\mathbf{a}} = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

$(b)$

$$C(\mathbf{A}^T) = span\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\} = span\{\mathbf{b}\}$$

$$\Rightarrow \text{P}_\text{R} = \frac{\mathbf{bb}^T}{\mathbf{b}^T\mathbf{b}} = \begin{bmatrix} 0.1111 & 0.2222 & 0.2222 \\ 0.2222 & 0.4444 & 0.4444 \\ 0.2222 & 0.4444 & 0.4444 \end{bmatrix}$$

$$\mathbf{B} = \mathbf{P_C A P_R} = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} = \mathbf{A}$$

$\mathbf{P_C A P_R}$

$= \mathbf{P_C}(\mathbf{P_R}^T\mathbf{A}^T)^T$

$= \mathbf{P_C}(\mathbf{P_R}\mathbf{A}^T)^T$ $\quad \mathbf{P_R}$ is symmetric

$= \mathbf{P_C}(\mathbf{A}^T)^T$ $\quad \mathbf{P_R}\mathbf{A}^T$ (project $\mathbf{A}^T$ onto C($\mathbf{A}^T$))

$= \mathbf{P_C}\mathbf{A}$ $\quad \mathbf{P_C}\mathbf{A}$ (project $\mathbf{A}$ onto C($\mathbf{A}$))

$= \mathbf{A}$