Solutions to Homework 8

1.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\mathbf{A^T A \hat{x} = A^T b}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\mathbf{p} = \mathbf{A}\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

$$\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$$

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = \mathbf{e}^T\mathbf{e} = 44$$

2.

The four points are $P_1(0\ ,\ 0)$, $P_2(1\ ,\ 8)$, $P_2(3\ ,\ 8)$ and $P_1(4\ ,\ 20)$.

Then substitute them into the equation which is the problem given.

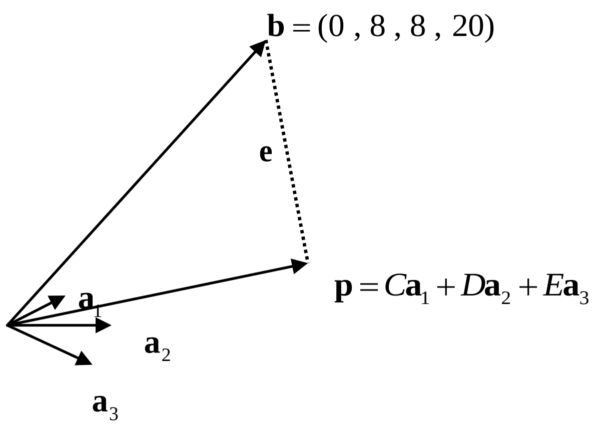
$$C+D\cdot 0+E\cdot 0\approx 0$$
$$C+D\cdot 1+E\cdot 1\approx 8$$
$$C+D\cdot 3+E\cdot 9\approx 20$$
$$C+D\cdot 4+E\cdot 16\approx 20$$

The matrix form is

$$\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 3 & 9\\ 1 & 4 & 16\end{bmatrix}\begin{bmatrix}C\\ D\\ E\end{bmatrix}\approx\begin{bmatrix}0\\ 8\\ 20\\ 20\end{bmatrix}\triangleq \mathbf{A}\hat{\mathbf{x}}\approx\mathbf{b}$$

So the normal equation is following

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{x}}=\mathbf{A}^{\mathrm{T}}\mathbf{b}$$



$\mathbf{a}_1=(1\ ,1\ ,1\ ,1)$

$\mathbf{a}_2=(0\ ,1\ ,3\ ,4)$

$\mathbf{a}_3=(0\ ,1\ ,9\ ,16)$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ will span $C(\mathbf{A})$, $\mathbf{p}$ is the linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
And $\mathbf{p}$ is also the projection of $\mathbf{b}$ onto $C(\mathbf{A})$.
The residual of $\mathbf{p}$ and $\mathbf{b}$ is $\mathbf{e}$, and $\mathbf{e}$ is perpendicular on $\mathbf{p}$.
$\mathbf{p} \in C(\mathbf{A})$, $\mathbf{e} \in$ left $N(\mathbf{A})$.

3.

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{u_1} = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \mathbf{q_1} = \frac{\mathbf{u_1}}{\|\mathbf{u_1}\|} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{a_2} - \frac{\mathbf{u_1}\mathbf{u_1}^{\mathbf{T}}}{\mathbf{u_1}^{\mathbf{T}}\mathbf{u_1}}\mathbf{a_2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{A = QR =} \begin{bmatrix} | & | \\ \mathbf{q_1} & \mathbf{q}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \mathbf{a_1}^{\mathbf{T}}\mathbf{q_1} & \mathbf{a}_2^{\mathbf{T}}\mathbf{q_1} \\ 0 & \mathbf{a}_2^{\mathbf{T}}\mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

To solve the least squares problem, recall the normal equation

$$\Rightarrow \mathbf{A^T A\hat{x} = A^T b}$$

$$\Rightarrow (\mathbf{QR})^{\mathbf{T}}\mathbf{QR\hat{x}} = (\mathbf{QR})^{\mathbf{T}}\mathbf{b}$$

$$\Rightarrow \mathbf{R^T Q^T QR\hat{x} = R^T Q^T b} \qquad (\mathbf{Q} \text{ is orthogonal } \therefore \mathbf{Q^T = Q^{-1}})$$

$$\Rightarrow \mathbf{R^T R\hat{x} = R^T Q^T b}$$

$$\Rightarrow \mathbf{R\hat{x} = Q^T b}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} 5/9 \\ 0 \end{bmatrix}$$

4.

The projection matrix is $\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$.

But the rank of $\mathbf{A}$ is 2 , $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ will not be invertible.

So we remove the dependent columns of A first and keep the independent columns.

Then a new matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$, the column space of new $\mathbf{A}$ and original $\mathbf{A}$ are the same.

$\therefore$ we can use the new $\mathbf{A}$ to get the projection matrix onto $C(\mathbf{A})$

$$\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\because$ left $N(\mathbf{A}) \perp C(\mathbf{A})$

$\therefore$ The projection matrix onto left $N(\mathbf{A}) = I - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$

To find the projection matrix onto row space of $\mathbf{A}$, we also remove the dependent rows and keep the independent rows , then the new $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

$\therefore$ The projection matrix onto $C(\mathbf{A}^{\mathrm{T}}) = \mathbf{A}^{\mathrm{T}}\left[(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\right]^{-1}(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}$

And $N(\mathbf{A}) \perp C(\mathbf{A}^{\mathrm{T}})$

$\therefore$ The projection matrix onto $N(\mathbf{A}) = I - \mathbf{A}^{\mathrm{T}}\left[(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\right]^{-1}(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \frac{1}{6}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

5.

(a)

$$\mathbf{Ax} = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

S is the nullspace of A and dim(S)=4-rank(A)=3

$$\Rightarrow \text{S=span}\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}$$

(b)

S is the nullspace of A.

$\therefore \text{S}^{\perp}$ is the row space of A.

$$\Rightarrow \text{S}^{\perp}\text{=span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\}$$

(c)

$$\mathbf{b_1} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 1.5 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

6.

$\mathbf{A} = [\mathbf{Q} \quad \mathbf{a}] = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n \quad \mathbf{a}]$

$\Rightarrow \{\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n\}$ are orthonormal columns

We use Gram-Schmidt method to orthonormalize $\mathbf{a}$

$\Rightarrow$

$$\begin{aligned}
\mathbf{u}_a &= \mathbf{a} - \left(\mathbf{a}^{\mathrm{T}}\mathbf{q}_1\right)\mathbf{q}_1 - \left(\mathbf{a}^{\mathrm{T}}\mathbf{q}_2\right)\mathbf{q}_2 - \cdots - \left(\mathbf{a}^{\mathrm{T}}\mathbf{q}_n\right)\mathbf{q}_n \\
&= \mathbf{a} - \left[\left(\mathbf{a}^{\mathrm{T}}\mathbf{q}_1\right)\mathbf{q}_1 + \left(\mathbf{a}^{\mathrm{T}}\mathbf{q}_2\right)\mathbf{q}_2 + \cdots + \left(\mathbf{a}^{\mathrm{T}}\mathbf{q}_n\right)\mathbf{q}_n\right] \\
&= \mathbf{a} - \left[\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n\right]\begin{bmatrix} \mathbf{q}_1^{\mathrm{T}}\mathbf{a} \\ \mathbf{q}_2^{\mathrm{T}}\mathbf{a} \\ \vdots \\ \mathbf{q}_n^{\mathrm{T}}\mathbf{a} \end{bmatrix} \\
&= \mathbf{a} - \left[\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n\right]\begin{bmatrix} \mathbf{q}_1^{\mathrm{T}} \\ \mathbf{q}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{q}_n^{\mathrm{T}} \end{bmatrix}\mathbf{a} \\
&= \mathbf{a} - \mathbf{Q}\mathbf{Q}^{\mathrm{T}}\mathbf{a} \\
&= \mathbf{a} - \mathbf{P}\mathbf{a}
\end{aligned}$$

Then , normalize $\mathbf{u}_a$

$$\Rightarrow \mathbf{q}_a = \frac{\mathbf{u}_a}{\|\mathbf{u}_a\|} = \frac{\mathbf{a} - \mathbf{P}\mathbf{a}}{\sqrt{(\mathbf{a} - \mathbf{P}\mathbf{a})^{\mathrm{T}}(\mathbf{a} - \mathbf{P}\mathbf{a})}}$$

$\therefore$ The new orthonormal matrix is $\left[\mathbf{Q} \quad \mathbf{q}_a\right]$