Solutions to Homework 9

1.

$$\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix}=\det\begin{bmatrix}101 & 201 & 301\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}=\det\begin{bmatrix}101 & 201 & 301\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}=0$$

$$\det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix}=\det\begin{bmatrix}1 & t & t^2\\ 0 & 1-t^2 & t-t^3\\ 0 & t-t^3 & 1-t^4\end{bmatrix}=(1-t^2)(1-t^4)-(t-t^3)(t-t^3)$$

$$=(1-t^2)^2(1+t^2)-t^2(1-t^2)^2=(1-t^2)^2$$

2.

$$G_2=\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}=0-1=-1$$

$$G_3=\begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix}=0+1+1-0-0-0=2$$

$$G_4=\begin{vmatrix}0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{vmatrix}\xrightarrow[\substack{R_{13}(-1)\\ R_{14}(-1)}]{R_{12}(-1)}\begin{vmatrix}0 & 1 & 1 & 1\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\end{vmatrix}\xrightarrow{R_{21}(1)}\begin{vmatrix}1 & 0 & 1 & 1\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\end{vmatrix}$$

$$\xrightarrow{R_{31}(1)}\begin{vmatrix}2 & 0 & 0 & 1\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\end{vmatrix}\xrightarrow{R_{41}(1)}\begin{vmatrix}3 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\end{vmatrix}=3\times(-1)^{1+1}\times\begin{vmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{vmatrix}=-3$$

$$G_n=\begin{vmatrix}0 & 1 & \cdots & 1\\ 1 & 0 & & \vdots\\ \vdots & & \ddots & 1\\ 1 & \cdots & 1 & 0\end{vmatrix}$$

We can predict G_n after doing row operation , G_n will be $\begin{vmatrix} n-1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & 0 \\ \vdots & & \ddots & \\ 1 & 0 & \cdots & -1 \end{vmatrix}$

$\therefore G_n = (n-1)\times(-1)^{1+1}\times|\mathbf{I}_{\text{n-1}}| = (n-1)\times(-1)^{n-1}$

3

(a)

A permutation matrix can be considered as an identity matrix after several row exchangings.

$\therefore$ det P = 1,-1

(b)

If P is a projection matrix

$\Rightarrow P^2 = P$

$\Rightarrow (\det P)^2 = \det P$

$\Rightarrow \det P = 0,1$

(c)

$A^T = -A$

$\Rightarrow \det(A^{\text{T}}) = \det(-A)$

$\Rightarrow \det A = -\det A$

$\Rightarrow$ det A=0

4.

Use the block determinants

$$\Rightarrow \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{CA}^{-1} & \mathbf{I} \end{vmatrix} = |\mathbf{I}|\times|\mathbf{I}| = 1$$

$$\Rightarrow \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{CA}^{-1} & \mathbf{I} \end{vmatrix}\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D-CA^{-1}B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix}$$

$$\therefore \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = |\mathbf{A}|\times|\mathbf{D-CA^{-1}B}| = |\mathbf{A}\times(\mathbf{D-CA^{-1}B})| = |\mathbf{AD-ACA^{-1}B}| \text{ , if } \mathbf{AC}=\mathbf{CA}$$

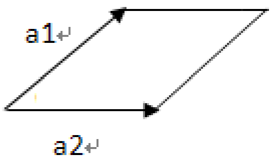
$$\Rightarrow \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = |\mathbf{AD-CB}|$$

5.

(a)

$$Volume = \left| \det \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \right| = 20$$

(b)



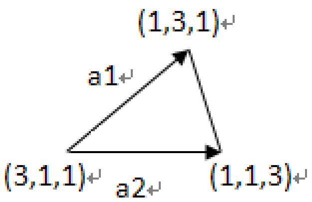
$$\mathbf{a}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow Area = \sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})} = 6\sqrt{2}$$

(http://ccjou.twbbs.org/blog/?p=7020)

(c)



$$\mathbf{a}_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow Area = \frac{1}{2}\sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})} = 2\sqrt{3}$$

(http://ccjou.twbbs.org/blog/?p=7020)

6.

We know all cofactors of A

$$\Rightarrow \text{Let a matrix } \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

According to the property of cofactor , we can get $\mathbf{AC}^{\mathrm{T}} = (\det\ \mathbf{A})\mathbf{I}$

$\Rightarrow (\det\ \mathbf{A})^{-1}\mathbf{AC}^{\mathrm{T}} = \mathbf{I}$

$\Rightarrow \mathbf{A}^{-1} = (\det\ \mathbf{A})^{-1}\mathbf{C}^{\mathrm{T}}$

Now we want to find $\det\ \mathbf{A}$

$$\Rightarrow \left|\mathbf{AC}^{\mathrm{T}}\right| = |\mathbf{A}|\left|\mathbf{C}^{\mathrm{T}}\right| = |\mathbf{A}||\mathbf{C}| = \begin{vmatrix} |\mathbf{A}| & 0 & 0 & 0 \\ 0 & |\mathbf{A}| & 0 & 0 \\ 0 & 0 & |\mathbf{A}| & 0 \\ 0 & 0 & 0 & |\mathbf{A}| \end{vmatrix} = (|\mathbf{A}|)^4$$

$$\Rightarrow |\mathbf{A}| = (|\mathbf{C}|)^{\frac{1}{3}}$$

$\therefore$ we can find $\mathbf{A}^{-1}$ then inverse it to get $\mathbf{A}$