Matrix Theory

Problem Set 10 **2010**

Due Monday, 3 January 2011 at 4:30 pm in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (20pts) Answer the following questions.
   (a) Prove that the eigenvalues of any $n \times n$ skew-Hermitian matrix are pure imaginary number.
   (b) Let A be an $n \times n$ skew-symmetric matrix, i.e., $A^T = -A$. Show that e^A is an orthogonal matrix, i.e., $(e^A)^{-1} = (e^A)^T$. If A is skew-Hermitian, is it true that e^A is an unitary matrix?
2. (15pts) Find the SVD of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   Based on the SVD you obtain, write down orthonormal bases for $C(A),\ C(A^T),\ N(A),\ N(A^T)$, respectively.
3. (20pts) Let A be an n by n matrices. Answer true or false.
   (a) If $\text{rank}A = 0$, then $A = 0$.
   (b) If $A^2 = 0$, then $A = 0$.
   (c) If $A^* A = 0$, then $A = 0$.
   (d) If all the eigenvalues of A are 1, then $A = I$.
   (e) If all the eigenvalues of A are 0, then $A = 0$.
   (f) If all the singular values of A are 0, then $A = 0$.
   (g) The number of nonzero eigenvalues of A equals $\text{rank}A$.
   (h) The number of nonzero singular values of A equals $\text{rank}A$.
   (i) Similar matrices have the same eigenvalues.
   (j) Similar matrices have the same singular values.
4. (15pts) If A is 5 by 4, and the nonzero singular values of A are 4, 3, 2, 1, show the value for each of the following functions:
   (1) $\det(A^T A)$
   (2) $\text{trace}(AA^T)$
   (3) $\dim N(AA^T)$
   (4) $\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$

(5) $\min_{\mathbf{x}\neq\mathbf{0}} \dfrac{\mathbf{x}^T A^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$

(6) $\max_{\mathbf{w}} \min_{\mathbf{x}\neq\mathbf{0},\mathbf{x}\perp\mathbf{w}} \dfrac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|^2}$

(7) $\min_{\mathbf{w}} \max_{\mathbf{x}\neq\mathbf{0},\mathbf{x}\perp\mathbf{w}} \dfrac{\|A^T\mathbf{x}\|}{\|\mathbf{x}\|}$

5. (15pts) Use SVD to answer the following questions.
   (a) Let A be an $n\times n$ real matrix. Prove that $A^T A$ is similar to AA^T.
   (b) Let A be an $n\times n$ real matrix. Find matrices B and C so that $B^2 = A^T A$ and $C^2 = AA^T$.
6. (15pts) Let A be an m by n matrix. Suppose the SVD of A is $A = U\Sigma V^T$. If $\operatorname{rank}A = n$, show that the least squares solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\hat{\mathbf{x}} = \sum_{i=1}^{r} \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}\mathbf{v}_i.$$