Matrix Theory

Problem Set 2 **2010**

Due Thursday, 7 October 2010 at 12 noon in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (20pts) The sum $W_1 + W_2$ of two subspaces W_1 and W_2 of a finite-dimensional vector space V is called a *direct sum*, denoted by $W_1 \oplus W_2$, if $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$, $\mathbf{w}_1 \in W_1$, $\mathbf{w}_2 \in W_2$ implies $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$, i.e, $W_1 \cap W_2 = \{\mathbf{o}\}$. Prove that $W_1 + W_2$ is a direct sum if and only if $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$. (Hint: start with bases for W_1 and W_2.)

2. (15pts) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ are bases for an n-dimensional vector space V, show that there exists an invertible linear transformation T on V such that $T(\mathbf{v}_j) = \mathbf{w}_j$, $j = 1, \ldots, n.$ (Hint: think of matrix representation.)

3. (15pts) Consider the vector space consisting of all real 2 by 2 matrices and let T be the linear transformation on this space that sends each matrix X onto PX, i.e, $T(X) = PX$, where $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the matrix representation of T with respect to the ordered basis consisting of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

4. (15pts) If $A = [a_{ij}]$ is an n by n matrix such that $a_{ii} = 0$, $i = 1, \ldots, n$, then there exist matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ such that $A = BC - CB$. (Hint: try diagonal matrix.)

5. (15pts) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show that if a 2 by 2 matrix B commutes with A, i.e, $AB = BA$, then there exists a polynomial $p(t)$ such that $B = p(A)$. Note that for each polynomial, there is a corresponding matrix polynomial. For example, the corresponding matrix polynomial of $p(t) = t^2 - 2t + 3$ is $p(A) = A^2 - 2A + 3I$, where I is the identity matrix.

6. (20pts) Suppose A is a 3 by 3 matrix of the form $A = I - 2\mathbf{u}\mathbf{u}^T$, where $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$, i.e., $\mathbf{u}^T\mathbf{u} = 1$. Find a matrix A such that $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Is it possible to find a matrix A with the above form such that $A\mathbf{z} = \mathbf{y}$, where $\mathbf{z} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$? State your reasoning.