Matrix Theory

Problem Set 3 **2010**

Due Thursday, 14 October 2010 at 12 noon in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (10pts) One way to compute the inverse of a square matrix A is to apply elementary row operations to $\begin{bmatrix} A & I \end{bmatrix}$ to get $\begin{bmatrix} I & E \end{bmatrix}$, where E is the product of applied elementary matrices. It is easy to see that $EA = I$ and thus $E = A^{-1}$. Use this approach to find the inverses of the following matrices:

   (a) $A = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix}$, where $a_1, a_2, \ldots, a_n$ are nonzero scalars.

   (b) $B = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

2. (10pts) Let $A+B$ be an invertible matrix. Prove that
$$A - A(A+B)^{-1}A = B - B(A+B)^{-1}B.$$

3. (15pts) Let A and B be $n \times n$ invertible matrices. Find the inverses of the following matrices.

   (a) $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} A & I_n \\ I_n & 0 \end{bmatrix}$

4. (20pts) Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 4 \\ 1 & 1 & 3 & 1 & 6 \\ 1 & 1 & 4 & 0 & 3 \\ 1 & 1 & 5 & 0 & 3 \end{bmatrix}$. Apply elementary row operations to $\begin{bmatrix} A & I \end{bmatrix}$ to get the reduced row echelon form $\begin{bmatrix} R & E \end{bmatrix}$. Then, identify bases for the column space, row space, nullspace, and left nullspace of *A*. Use pivots to explain why the dimension of the column space is equal to the dimension of the row space, and that is why we call it the rank of *A*.

5. (20pts) Let *A* and *B* be $n \times n$ matrices.
   (a) Apply elementary operations to $\begin{bmatrix} I & A \\ B & I \end{bmatrix}$ to get $\begin{bmatrix} I - AB & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & I - BA \end{bmatrix}$. Then, conclude that $\operatorname{rank}(I - AB) = \operatorname{rank}(I - BA)$.
   (b) Apply elementary operations to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ to get $\begin{bmatrix} A + B & B \\ B & B \end{bmatrix}$ and then derive the following inequality: $\operatorname{rank}(A + B) \le \operatorname{rank} A + \operatorname{rank} B$.

6. (25pts) MATLAB Problem. An *n* by *n* 0-1 matrix has entries 0 or 1. Use the random number generator to produce random matrices. Answer the following questions.
   (a) What are the frequencies of different ranks? Draw the histogram.
   (b) What is the largest possible rank? That is, what is the most frequently occurring rank?

Start with *n*=2, and increase *n* gradually and see how your answers in (a), (b) change with *n*. Write down your findings in this computer experiment.