Matrix Theory

Problem Set 4 **2010**

Due <u>Monday, 25 October 2010</u> at 12 noon in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (15pts) Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be the subspaces in $\mathbb{R}^4$, where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -7 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -6 \\ -5 \end{bmatrix} \text{ and } \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the dimensions and bases for $V \cap W$ and $V + W$.

2. (15pts) Let A be 5×7 and B be 7×6 matrices.
   (a) If $\text{rank}(A) = 4$, find all possible values of $\text{rank}(AB)$.
   (b) If $\text{rank}(A) = 3$ and $\text{rank}(AB) = 2$, find all possible values of $\text{rank}(B)$.
   (c) Is it possible that $\text{rank}(A) = 3$, $\text{rank}(B) = 5$, and $AB=0$?

3. (20pts) Show that the intersection of three 6-dimensional subspaces in $\mathbb{R}^8$ is not the single point $\{\mathbf{0}\}$.

4. (20pts) Let A be m by n, and $\text{rank}(A) = r$. Let $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$ be an invertible matrix such that $EA = \begin{bmatrix} R \\ 0 \end{bmatrix}$, where R is r by n, and R is in reduced row echelon form. Show that $C(A) = N(E_2)$, where $C(A)$ is the column space of A, and $N(E_2)$ is the nullspace of E_2.

5. (15pts) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are bases of $\mathbb{R}^n$, and for every j,

$$A\mathbf{u}_j = c_{1j}\mathbf{u}_1 + c_{2j}\mathbf{u}_2 + \cdots + c_{nj}\mathbf{u}_n$$

$$B\mathbf{v}_j = c_{1j}\mathbf{v}_1 + c_{2j}\mathbf{v}_2 + \cdots + c_{nj}\mathbf{v}_n$$

Prove that there exists an invertible matrix S such that $A = SBS^{-1}$. Show the invertible matrix S you found.

6. (15pts) Suppose U and W are subspaces in a vector space V. If U and W are invariant subspaces under linear transformation A, prove that $U+W$ is an invariant subspace under A. Is $U \cap W$ also invariant under A? State your reasoning. Recall that a subspace S is invariant under linear transformation A if

$A(S) \triangleq \{A(\mathbf{x}) | \mathbf{x} \in S\} \subseteq S$.