Matrix Theory

Problem Set 5 **2010**

Due Thursday, 11 November 2010 at 12 noon in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (20pts) Let A be an $n \times n$ matrix.
   (a) If A is singular, what are the possible values of $\text{rank}(\text{adj}A)$? Why?
   (b) If A is singular and $n \geq 3$, use the results in (a) to prove that $\text{adj}(\text{adj}A) = 0$.
   (c) If S is invertible, show that $\text{adj}(SAS^{-1}) = S(\text{adj}A)S^{-1}$.
2. (15pts) For each of the following matrices, find all the eigenvalues and the corresponding eigenvectors.

$$A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}.$$

3. (15pts) Consider the block diagonal matrix $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where B is $m \times m$ and C is $n \times n$. Express the eigenvectors of A in terms of those of B and C. Also, show that the eigenvalues of A are those of B together with those of C.
4. (20pts) If $\mathbf{u} = (1,1,1,1,1)$, and $A = I + \mathbf{u}\mathbf{u}^T$, answer the following questions:
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Is A diagonalizable? If yes, express A as $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix.
   (c) What is the determinant of A?
   (d) What is the trace of A^2?
5. (15pts) Let $q(t) = a_0 + a_1 t + \cdots + a_m t^m$ be a given polynomial. The matrix polynomial of A is defined to be $q(A) = a_0 I + a_1 A + \cdots + a_m A^m$.
   (a) If $A\mathbf{x} = \lambda\mathbf{x}$, show $q(\lambda)$ is an eigenvalue of the matrix $q(A)$ and $\mathbf{x}$ is an eigenvector corresponding to $q(\lambda)$.
   (b) If A is diagonalizable, show that $q(A)$ is diagonalizable.
   (c) If A is similar to B, show that $q(A)$ is similar to $q(B)$.
6. (15pts) If A is diagonalizable, consider the characteristic polynomial $p_A(t)$ and show that $p_A(A)$ is the zero matrix. This is called Cayley-Hamilton theorem.