Matrix Theory

Problem Set 7 **2010**

Due Thursday, 2 December 2010 at 12 noon in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (10pts) Let the Jordan form of A be

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \oplus \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \oplus \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \oplus \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \oplus [3]$$

   (a) What is the size of A?
   (b) What is the rank of A?
   (c) What are the eigenvalues of A?
   (d) For each distinct eigenvalue, what are the corresponding algebraic multiplicities and geometric multiplicities?
   (e) Find all the independent eigenvectors of J.
   (f) What is the trace of A?

2. (20pts) Suppose U is an n by n unitary matrix, i.e., $U^* = U^{-1}$.
   (a) Show that the eigenvalues of U are all equal to 1 in absolute value.
   (b) Show that if λ is an eigenvalue of U, then $\frac{1}{\lambda}$ is an eigenvalue of U^*.
   (c) What are the possible values of $|\det U|$?
   (d) Show that the column vectors of U form an orthonormal basis for $\mathbb{C}^n$.
   (e) If A is similar to U, show that A^* is similar to A^{-1}.

3. (20pts) Suppose A is a normal matrix, i.e., $A^*A = AA^*$.
   (a) Show that $I - A$ is normal.
   (b) Show that $\|A\mathbf{x}\| = \|A^*\mathbf{x}\|$ for every complex vector $\mathbf{x}$. Note that

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^*(A\mathbf{x}).$$

   (c) Use the result in (b) to show that $\|(A - \lambda I)\mathbf{x}\| = \|(A^* - \bar{\lambda} I)\mathbf{x}\|$.
   (d) Use the result in (c) to show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^*\mathbf{x} = \bar{\lambda}\mathbf{x}$.
   (e) Show that the eigenvectors belonging to distinct eigenvalues are orthogonal.

4. (10pts) Let A be a square matrix. Define $H = \frac{1}{2}\left(A + A^*\right)$, the Hermitian part, and $S = \frac{1}{2}\left(A - A^*\right)$, the skew-Hermitian part, of A. Then, $A = H + S$. Show that A is normal if and only if $HS = SH$.

5. (15pts) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

   (a) Find $\mathbf{x}$ so that $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\|\mathbf{x}\|$ is minimum.

   (b) Among all $\mathbf{x}$ that minimizes $\left\| A\mathbf{x} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\|$, find the one with the minimum length $\|\mathbf{x}\|$. (Hint: First look for a nice basis for the column space of A.)

6. (25pts) Let $W = \operatorname{span}\{1, x\} \subset P_2$, where P_2 is a vector space of polynomial of degree at most 2 and has standard inner product for $p, q \in P_2$:

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

   (a) Use Gram-Schmidt process to find an orthonormal basis for W.
   (b) Find the least-squares error approximation to of x^3 on the interval [0,1] by a function in W.
   (c) Compute the orthogonal complement $W^\perp$ by showing a basis of it.