Matrix Theory

Problem Set 9 **2010**

Due Thursday, 23 December 2010 at 12 noon in EE105. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print every group member's name and student ID on your homework.

1. (15pts) Explain what is wrong with the following arguments for $p_A(A)=0$.
   (a) "Since $p_A(\lambda)=0$ for every eigenvalue λ of A, and since the eigenvalues of $q(A)$, q is a polynomial, are the $q(\lambda)$, it follows that all eigenvalues of $p_A(A)$ are 0. Therefore, $p_A(A)=0$."
   (b) "Since $p_A(t)=\det(tI-A)$, $p_A(A)=\det(AI-A)=\det(A-A)=\det 0=0$. Therefore, $p_A(A)=0$."
2. (15pts) Find the characteristic polynomial and the minimal polynomial of the following matrix:
$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 1 & 1 & 3 & 4 \end{bmatrix}.$$
   Is this matrix diagonalizable? What is its Jordan form?
3. (10pts) Answer the following questions.
   (a) Find a 2×2 unitary and Hermitian matrix A, i.e., $A^*=A^{-1}, A^*=A$.
   (b) Find a 2×2 unitary and skew-Hermitian matrix B, i.e, $B^*=B^{-1}, B^*=-B$.
4. (20pts) Let A and B be n by n Hermitian matrices. Answer true or false.
   (a) $A+B$ is Hermitian.
   (b) cA is Hermitian for every complex number c.
   (c) AB is Hermitian.
   (d) ABA is Hermitian.
   (e) If $A^2=I$, then $A=I$.
   (f) If $A^2=0$, then $A=0$.
   (g) iA is skew-Hermitian.
   (h) The diagonal entries of A are all real.
5. (15pts) Answer the following questions. Let A and B be n by n matrices.
   (a) Show that if A and B are Hermitian and $AB=BA$, then AB is Hermitian.
   (b) If A is Hermitian and A is similar to B, is B also Hermitian? Prove it or give a counter example.
6. (25pts) Answer the following questions.
   (a) Find the values of a and b so that the following matrices are positive

semidefinite.

$$\begin{bmatrix} 1 & 1 & a \\ 1 & 1 & 1 \\ \bar{a} & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b & 0 \\ \bar{b} & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) If A is Hermitian, show that A^2 is positive semidefinite.

(c) If A is skew-Hermitian, show that $-A^2$ is positive semidefinite.

(d) If A is skew-Hermitian, is it true that for every $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^* A\mathbf{x} = 0$? State your reasoning.

(e) If $\mathbf{x}^* A\mathbf{x} = 0$ for some $\mathbf{x} \neq 0$, does it follow that $A = 0$ or $A\mathbf{x} = \mathbf{0}$? What if A is positive semidefinite?