Solutions to problem set 1

1.

(a) ANS: True

(b) ANS: False

let $W=\{(x,y)\mid x^2+y^2=0, x,y\in\mathbb{C}\}$

$x^2+y^2=0 \Rightarrow x=\pm yi$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ i \end{pmatrix} , \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}\in W , \begin{pmatrix} 1 \\ -i \end{pmatrix}\in W$$

but $\begin{pmatrix} 1 \\ i \end{pmatrix}+\begin{pmatrix} 1 \\ -i \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}\notin W$

$\Rightarrow W=\{(x,y)\mid x^2+y^2=0, x,y\in\mathbb{C}\}$ is not a subspace of $\mathbb{C}^2$

(c) ANS: False

let $W=\{(x,y)\mid x^2+y^2=1, x,y\in\mathbb{R}\}$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}\in W , \begin{pmatrix} 0 \\ 1 \end{pmatrix}\in W$$

but $\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}\notin W$

$\Rightarrow W=\{(x,y)\mid x^2+y^2=1, x,y\in\mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$

(d) ANS: False

let $W=\{(x,y)\mid x-y=1, x,y\in\mathbb{R}\}$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}\in W , \begin{pmatrix} 0 \\ -1 \end{pmatrix}\in W$$

but $\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}\notin W$

$\Rightarrow W=\{(x,y)\mid x+y=1, x,y\in\mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$

(e) ANS: True

(f) ANS: True

(g)ANS: False

let $W = \{p(t) \mid p(t) \in P_n \ , \ p(t) \ \text{has degree} \ 2\}$

let $p_1(t) \in W \ , \ p_2(t) \in W$

$p_1(t) = -a \cdot t^2$

$p_2(t) = a \cdot t^2$

$\Rightarrow p_1(t) + p_2(t) = 0 \cdot t^2$

$\Rightarrow p_1(t) + p_2(t) \notin W$

$\Rightarrow W = \{p(t) \mid p(t) \in P_n \ , \ p(t) \ \text{has degree} \ 2\}$ is not a subspace of P_n

(h)ANS: True

(i)ANS: False

let $W = \{p(t) \mid p(t) \geq 0, p(t) \in P_n\}$

let $x(t) \in W$

assume $c < 0$

$\Rightarrow c \cdot x(t) \notin W$

$\Rightarrow W = \{p(t) \mid p(t) \geq 0, p(t) \in P_n\}$ is not a subspace of P_n

2.

(a)

(1) prove that $W_1 + W_2$ is subspace

$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$

let $w_{X1} \in W_1 \ , \ w_{X2} \in W_2 \Rightarrow w_{X1} + w_{X2} \in W_1 + W_2$

let $w_{Y1} \in W_1 \ , \ w_{Y2} \in W_2 \Rightarrow w_{Y1} + w_{Y2} \in W_1 + W_2$

$\Rightarrow (w_{X1} + w_{X2}) + (w_{Y1} + w_{Y2}) = (w_{X1} + w_{Y1}) + (w_{X2} + w_{Y2}) \in W_1 + W_2$

let c is a constant

$c \cdot (w_{X1} + w_{X2}) \in W_1 + W_2$

$\Rightarrow W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$ is a subspace of V.

(2) prove that $W_1 \cap W_2$ is subspace

let $x \in W_1 \cap W_2$, $y \in W_1 \cap W_2$

$\Rightarrow x \in W_1$ and $x \in W_2$, $y \in W_1$ and $y \in W_2$

$\Rightarrow x + y \in W_1$ ($\because W_1$ is a subspace) and $x + y \in W_2$ ($\because W_2$ is a subspace)

$\Rightarrow x + y \in W_1 \cap W_2$

$x \in W_1 \cap W_2$

$\Rightarrow x \in W_1$ and $x \in W_2$

let c is a constant

$\Rightarrow c \cdot x \in W_1$ ($\because W_1$ is a subspace) and $c \cdot x \in W_2$ ($\because W_2$ is a subspace)

$\Rightarrow c \cdot x \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ is a subspace

(3) prove that $W_1 \cap W_2 \subseteq W_1 \cup W_2$

let $\forall x \in W_1 \cap W_2$

$\Rightarrow x \in W_1$ and $x \in W_2$

$\Rightarrow x \in W_1 \cup W_2$

$\Rightarrow W_1 \cap W_2 \subseteq W_1 \cup W_2$

(4) prove that $W_1 \cup W_2 \subseteq W_1 + W_2$

let $\forall x \in W_1 \cup W_2$

$\Rightarrow x \in W_1$ or $x \in W_2$

$\because W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$

$\therefore x \in W_1 + W_2$

$\Rightarrow W_1 \cup W_2 \subseteq W_1 + W_2$

by (3) and (4)

$\Rightarrow W_1 \cap W_2 \subseteq W_1 \cup W_2 \subseteq W_1 + W_2$

(b)

let $W_1 = \{(x,0) \mid x \in \mathbb{R}\}$

$W_2 = \{(0,y) \mid y \in \mathbb{R}\}$

$\Rightarrow W_1$ is x axis. W_2 is y axis.

let $w_1 = (a,0) \in W_1$, $w_2 = (0,b) \in W_2$ and $a,b \neq 0$

$w_1, w_2 \in W_1 \cup W_2$

$\Rightarrow w_1 + w_2 = (a,b)$

$\Rightarrow w_1 + w_2 \notin W_1 \cup W_2$

$\Rightarrow W_1 \cup W_2$ is not a subspace.

(c)

when $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

3

(a)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ -a-b-c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$W = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \}$$

$\Rightarrow \dim(W) = 3$

(b)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$W = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$$

$\Rightarrow \dim(W) = 3$

(c)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ a-b \\ a+b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$W = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \}$$

$$\Rightarrow \dim(W) = 2$$

(d)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$W = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}$$

$$\Rightarrow \dim(W) = 1$$

4.

(a)

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-3y\\y+z\\0\end{pmatrix}=x\begin{pmatrix}1\\0\\0\end{pmatrix}+y\begin{pmatrix}-3\\1\\0\end{pmatrix}+z\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\because\begin{pmatrix}-3\\1\\0\end{pmatrix}=(-3)\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Rightarrow \text{the base of } \mathrm{Ran}(T)=\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\}$$

$$\Rightarrow \dim(\mathrm{Ran}(T))=2$$

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-3y\\y+z\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\because \text{For any constant } c,\ \begin{pmatrix}x\\y\\z\end{pmatrix}=c\begin{pmatrix}3\\1\\-1\end{pmatrix} \text{ satisfies } T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\text{the base of } \mathrm{Ker}(T) \text{ is } \{\begin{pmatrix}3\\1\\-1\end{pmatrix}\}$$

$$\Rightarrow \dim(\mathrm{Ker}(T))=1$$

(b)

$$\dim(\mathrm{Ran}(L))=2$$

$$\Rightarrow \dim(\mathrm{Ker}(L))=\dim(\mathbb{R}^3)-\dim(\mathrm{Ran}(L))$$

$$\Rightarrow \dim(\mathrm{Ker}(L))=1$$

$$\Rightarrow \dim(\mathrm{Ran}(L))>\dim(\mathrm{Ker}(L))$$

$$\Rightarrow \mathrm{Ran}(L)\not\subset \mathrm{Ker}(L)$$

$$\Rightarrow \text{It's impossible to find } L \text{ such that } \mathrm{Ran}(L)\subseteq \mathrm{Ker}(L)$$

5.

＊ Main.m：

```
function [fraction, max_determinant] = Main(n, numbers)
% For a specified size "n", generate "numbers" square binary matrices,
%   then calculate the fraction of singular matrices, and find out
%   the maximum determinant of all possible determinants.
% After calculation, plot the histogram of the determinants.

% Input parameters:
%   n: size of the square matrix
%   numbers: how many matrices user want to generate

% Output parameter:
%   fraction: # of singular matrices / total # of square matrices
%   max_det: maximum determinant of all possible determinants


 [determinant_vector, fraction, max_determinant] = Calculate(n,
numbers);
 PlotDet(determinant_vector);

end
```

＊ Trend.m：

```
function Trend(max_size, t)
% For user-defined maximum size of square matrix, find out the trend of
%   "fraction" and "maximum determinant" when matrix size increases from
%   1 to user-defined maximum size max_size, and then plot them.

% Input parameter:
%   max_size: maximum size of the square matrix
%   t: number of times that Calculate.m being executed

avg_frac(max_size) = 0.0;
max_determinant(max_size) = 0.0;
fraction(max_size,t)=0;
```

```
max_det(max_size,t)=0;

for i = 1:max_size     % increase the size from 1 to max_size
    for j = 1:t        % for each size, execute Calculate.m "t" times
        [m,fraction(i,j),max_det(i,j)]= Calculate(i,10000);
        % generate 10000 square matrices of size n for each execution
        avg_frac(i) = avg_frac(i) + fraction(i,j);
    end
    max_determinant(i) = max(max_det(i));
    avg_frac(i) = avg_frac(i)/t;
end
subplot(2,1,1);
bar(avg_frac);
xlabel([{'n'},{'Size of Square Matrix'}]);
ylabel('Fraction of Singular Matrix');
title('Average Fraction for each Size');
subplot(2,1,2);
bar(max_determinant);
xlabel([{'n'},{'Size of Square Matrix'}]);
ylabel('Maximum Determinant');
title('Maximum Determinant for each Size');
end
```

＊ Calculate.m：

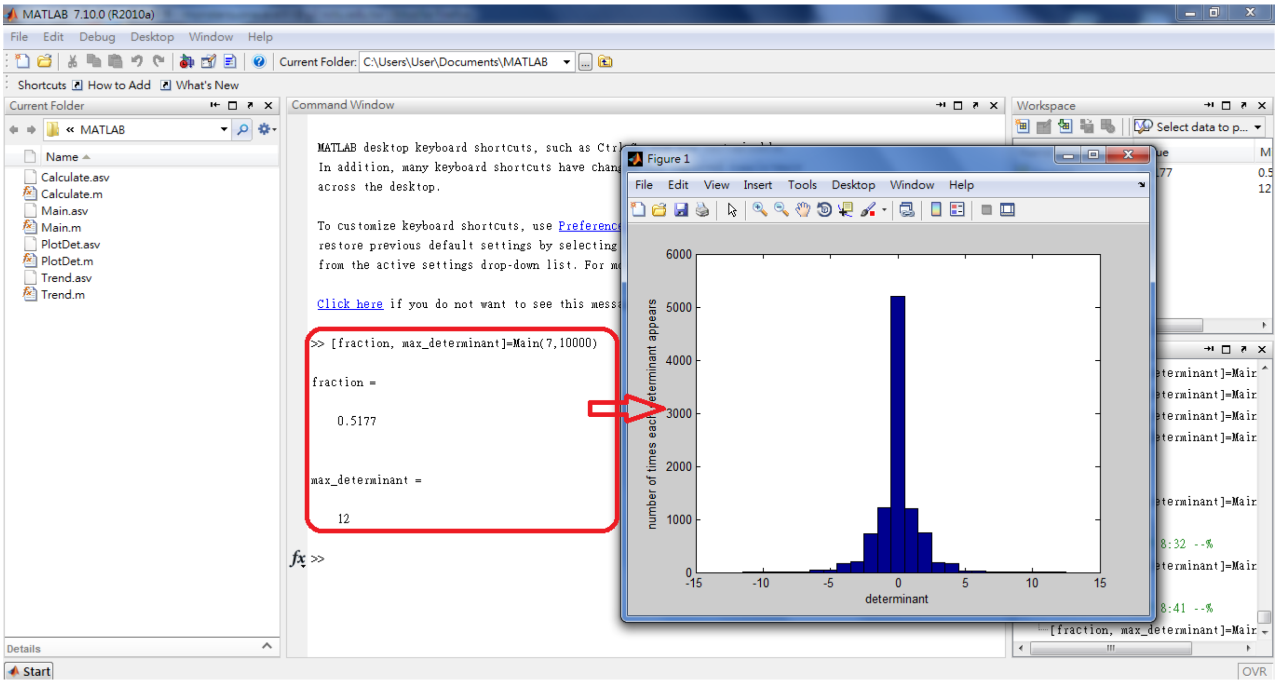
```
function [m, fraction, max_det]  = Calculate( n, numbers )
% For a specified size "n", generate "numbers" square binary matrices,
%    then calculate the fraction of singular matrices, and find out
%    the maximum determinant from the vector m

% Input parameters:
%    n: size of the square matrix
%    numbers: how many matrices user want to generate

% Output parameter:
%    m: vector that records every determinant values
%    fraction: # of singular matrices / total # of square matrices
```

```
%   max_det: maximum determinant of all possible determinants
count = 0;
for i = 1:numbers
    m(i) = det(round(rand(n)));
    if m(i) == 0    % singular
        count = count + 1;
    end
end
fraction = count / numbers;
max_det = max(m);
end
```

＊ PlotDet.m：

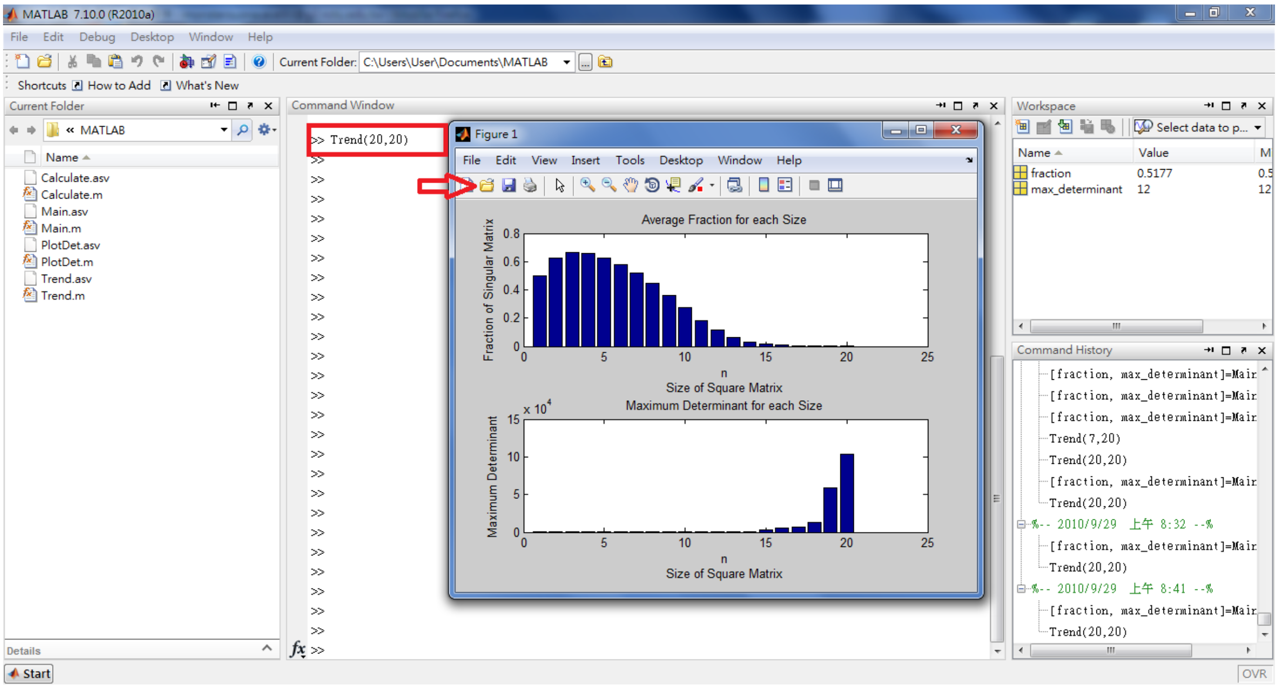
```
function PlotDet(determinant_vector)
% Plot the frequency of each determinant.

% Input parameter:
%   determinant_vector: vector that records every determinant values

close all;
x=[min(determinant_vector):max(determinant_vector)];
hist(determinant_vector,x);
xlabel('determinant');
ylabel('number of times each determinant appears');

end
```

＊ Execution of Main.m：



(picture 1)

＊ Execution of Trend.m：



(picture2)

(a) By picture 2 , when n increases , the fraction of singular matrix decreases and approaches to zero.

(b) By picture 1 , the determinant which is close to 0 has the most occurring probability.

(c) " det(A)=0 " has the largest frequency.

Remark：

By picture 2, the maximum determinant increases significantly as n increases.

＊For an n by n binary matrix A ,the determinant of A satisfies

$$|\det(A)| \le \frac{(n+1)^{\frac{n+1}{2}}}{2^n}$$

(Ref : http://mathworld.wolfram.com/HadamardsMaximumDeterminantProblem.html)