Solution to problem set 10

#1 (20pts)

(a)

let A is a $n\times n$ skew-Hermitian matrix

$Ax=\lambda x$

$\Rightarrow x^*Ax=\lambda\|x\|^2$

$\Rightarrow (x^*Ax)^* = \bar{\lambda}\|x\|^2$

$\Rightarrow (x^*Ax)^* = -x^*Ax = -x^*(\lambda x) = -\lambda\|x\|^2$

$\Rightarrow \bar{\lambda}\|x\|^2 = -\lambda\|x\|^2$

$\Rightarrow \bar{\lambda} = -\lambda$

$\Rightarrow$ the eigenvalues of any $n\times n$ skew-Hermitian matrix are pure imaginary number.

(b)

1.

$(e^A)^T = e^{A^T} = e^{-A}$

$(e^A)^T \times e^A = e^{-A} \times e^A$

$\because (-A)\times(A) = (A)\times(-A)$

$\therefore e^{-A}\times e^A = e^{(-A)+A} = I \Rightarrow e^{-A} = (e^A)^{-1}$

$\Rightarrow (e^A)^T = e^{-A} = (e^A)^{-1}$

$\Rightarrow e^A$ is an orthogonal matrix

2.

if A is skew-Hermitian

$(e^A)^* = e^{A^*} = e^{-A} = (e^A)^{-1}$

$\Rightarrow e^A$ is unitary matrix

#2 (15pts)

step1: find the eigenvalues and eigenvectors of A^TA, and the singular value of A

$$\lambda_1 = 2 \Rightarrow v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \sigma_1 = \sqrt{2}$$

$$\lambda_2 = 2 \Rightarrow v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \sigma_2 = \sqrt{2}$$

$$\lambda_3 = 0 \Rightarrow v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \sigma_3 = 0$$

$$\lambda_4 = 0 \Rightarrow v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \sigma_4 = 0$$

step2: Set up V and $\sum$

$$V = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sum = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

stept3: Build U

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

let $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$ by Gram–Schmidt orthogonalization

$$\Rightarrow u_3 = e_2 - \frac{\langle e_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle e_2, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$A=\begin{pmatrix}1&1&0&0\\0&0&1&0\\0&0&1&0\end{pmatrix}=U\sum V^T=\begin{pmatrix}0&1&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2}&0&0&0\\0&\sqrt{2}&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\1&0&0&0\\0&0&0&1\end{pmatrix}^T$$

the orthonormal basis of $C(A)=\{u_1,u_2\}=\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\}$

the orthonormal basis of $C(A^T)=\{v_1,v_2\}=\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\}$

the orthonormal basis of $N(A)=\{v_3,v_4\}=\{\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}$

the orthonormal basis of $N(A^T)=\{u_3\}=\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}\}$

#3 (20pts)

(a)True

(b)False

counter example: let $A=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}\neq 0\Rightarrow A^2=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}\begin{pmatrix}1&-1\\1&-1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

(c)True

(d)False

counter example: let $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}\neq I\Rightarrow$ eigenvalue of $A=1,1$

(f)True

pf : if all the singular values of A are 0 , than rankA=0 $\Rightarrow$ A=0.

(g)False

counter example: let $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}\Rightarrow$ eigenvalue of $A=0,0\Rightarrow$ number of nonzero eigenvalue $=0$

but $\text{rank}(A)=1$

(h)True

(i)True

(j)False

counter example : let $A=\begin{pmatrix}1 & 2\\ 0 & 4\end{pmatrix}$, $B=\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix}$

$\begin{pmatrix}1 & 2\\ 0 & 4\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}^{-1} \Rightarrow$ A and B are similar

singular values of A are 4.4954 and 0.8898

singular values of B are 5.0368 and 0.7942

$\Rightarrow$ sigular values of A $\neq$ singular values of B

#4 (15pts)

(1) $\det(A^TA)=\sigma_1{}^2\sigma_2{}^2\sigma_3{}^2\sigma_4{}^2=4^2\times3^2\times2^2\times1^2=576$

(2) $\mathrm{trace}(AA^T)=\mathrm{trace}(A^TA)=\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2+\sigma_4{}^2=4^2+3^2+2^2+1^2=30$

(3) the size of AA^T is 5×5

$\mathrm{rank}(AA^T)=\mathrm{rank}(A^TA)=\mathrm{rank}(A)=\#$ of nonzero singular value $=4$

by rank – nullity theorem: $\dim[N(AA^T)]=5-\mathrm{rank}(AA^T)=5-4=1$

(4) $\max_{\|x\|=1}\|Ax\|^2=\max_{\|x\|=1}x^TA^TAx=\lambda_{max}(x^Tx)=\lambda_{max}\|x\|^2=\lambda_{max}=\sigma_{max}{}^2=16$

$\Rightarrow \max_{\|x\|=1}\|Ax\|=\sqrt{16}=4$

(5) $\min_{x\neq0}\dfrac{x^TA^TAx}{x^Tx}=\lambda_{min}=\sigma_{min}{}^2=1$

(6) $\max_W \min_{\substack{x\neq0\\ x\perp W}}\dfrac{\|Ax\|^2}{\|x\|^2}=\sigma_3^2=2^2=4$

(7) $\because$ A and A^T have the same nonzero singular value

$\therefore \min_W \max_{\substack{x\neq0\\ x\perp W}}\dfrac{\|A^Tx\|}{\|x\|}=\sigma_2=3$

#5 (15pts)

(a)

$$\text{let } A=U\sum V^T \ , \ \sum=\begin{pmatrix}\sigma_1 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & \cdots & 0\\ \vdots & 0 & \sigma_r & \ddots & \vdots\\ 0 & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}$$

$\because$ the size of A is $n\times n \Rightarrow$ the size of $\sum$ is $n\times n \Rightarrow \sum^T=\sum$

$$\therefore \sum\sum^T=\sum^T\sum=\sum\sum=\begin{pmatrix}\sigma_1^2 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & \cdots & 0\\ \vdots & 0 & \sigma_r^2 & \ddots & \vdots\\ 0 & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}$$

$$A^TA=(U\sum V^T)^T(U\sum V^T)=(V\sum{}^TU^T)(U\sum V^T)=V\sum{}^T\sum V^T$$

$$\Rightarrow A^TA=V\begin{pmatrix}\sigma_1^2 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & \cdots & 0\\ \vdots & 0 & \sigma_r^2 & \ddots & \vdots\\ 0 & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}V^T \Rightarrow \sum{}^T\sum=\begin{pmatrix}\sigma_1^2 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & \cdots & 0\\ \vdots & 0 & \sigma_r^2 & \ddots & \vdots\\ 0 & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}=V^{-1}A^TA(V^T)^{-1}$$

$$AA^T=(U\sum V^T)(U\sum V^T)^T=(U\sum V^T)(V\sum{}^TU^T)=U\sum\sum{}^TU^T$$

$$\Rightarrow AA^T=U\begin{pmatrix}\sigma_1^2 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & \cdots & 0\\ \vdots & 0 & \sigma_r^2 & \ddots & \vdots\\ 0 & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}U^T \Rightarrow \sum\sum{}^T=\begin{pmatrix}\sigma_1^2 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & \cdots & 0\\ \vdots & 0 & \sigma_r^2 & \ddots & \vdots\\ 0 & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}=U^{-1}AA^T(U^T)^{-1}$$

$$\Rightarrow U^{-1}AA^T(U^T)^{-1}=\sum\sum{}^T=\sum{}^T\sum=V^{-1}A^TA(V^T)^{-1}$$

$$\Rightarrow AA^T=UV^{-1}A^TA(V^T)^{-1}U^T=UV^{-1}A^TA(UV^{-1})^T=UV^{-1}A^TA(UV^{-1})^{-1}$$

$\Rightarrow A^TA$ is similar to AA^T

(b)

$$B^2=A^TA=V\sum{}^T\sum V^T=V\sum{}^2V^T\Rightarrow B=V\sum V^T$$

$$C^2=AA^T=U\sum\sum{}^TU^T=U\sum{}^2U^T\Rightarrow C=U\sum U^T$$

#6 (15pts)

$$\hat{x}=(A^TA)^{-1}A^Tb$$

$$\Rightarrow \hat{x}=[(U\sum V^T)^T(U\sum V^T)]^{-1}(U\sum V^T)^Tb$$

$$\Rightarrow \hat{x}=[V\sum{}^TU^TU\sum V^T]^{-1}(V\sum{}^TU^T)b$$

$$\Rightarrow \hat{x}=[V\sum{}^T\sum V^T]^{-1}(V\sum{}^TU^T)b$$

$$\Rightarrow \hat{x}=[V(\sum{}^T\sum)^{-1}V^T](V\sum{}^TU^T)b$$

$$\Rightarrow \hat{x}=V(\sum{}^T\sum)^{-1}\sum{}^TU^Tb$$

$$\Rightarrow \hat{x}=\begin{pmatrix}| & & |\\ v_1 & \cdots & v_n\\ | & & |\end{pmatrix}\left(\begin{pmatrix}\sigma_1 & 0 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & 0 & \cdots & 0\\ 0 & 0 & \sigma_r & 0 & \ddots & \vdots\\ \vdots & 0 & 0 & 0 & \ddots & 0\\ 0 & \vdots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\end{pmatrix}_{n\times m}\begin{pmatrix}\sigma_1 & 0 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & 0 & \cdots & 0\\ 0 & 0 & \sigma_r & 0 & \ddots & \vdots\\ \vdots & 0 & 0 & 0 & \ddots & 0\\ 0 & \vdots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\end{pmatrix}_{m\times n}\right)^{-1}\begin{pmatrix}\sigma_1 & 0 & 0 & \cdots & 0 & 0\\ 0 & \ddots & 0 & 0 & \cdots & 0\\ 0 & 0 & \sigma_r & 0 & \ddots & \vdots\\ \vdots & 0 & 0 & 0 & \ddots & 0\\ 0 & \vdots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\end{pmatrix}_{n\times m}\begin{pmatrix}- & u_1 & -\\ - & u_2 & -\\ & \vdots & \\ - & u_m & -\end{pmatrix}b$$

$$\Rightarrow \hat{x} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1{}^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_r{}^2 & 0 & \ddots & \vdots \\ \vdots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{n\times n}^{-1} \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_r & 0 & \ddots & \vdots \\ \vdots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{n\times m} \begin{pmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_m & - \end{pmatrix} b$$

$$= \sum_{i=1}^{r} \frac{u_i{}^T b}{\sigma_i} v_i$$