Solutions to problem set 2

#1

step1 : prove that W_1+W_2 is a direct sum $\Rightarrow$ $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)$

let $U=\{u_1,u_2,...,u_m\}$ is the base of W_1

let $V=\{v_1,v_2,...,v_n\}$ is the base of W_2

let $w_1 \in W_1$, $w_2 \in W_2$

$\Rightarrow w_1 = c_1u_1 + c_2u_2 + ... + c_mu_m$

$\Rightarrow w_2 = d_1v_1 + d_2v_2 + ... + d_nv_n$

$w_1 + w_2 \in W_1 + W_2 = \text{span}\{u_1,u_2,...,u_m, v_1,v_2,...,v_n\}$

let $w_1 + w_2 = 0$

$\Rightarrow w_1 + w_2 = 0$ has unique solution : $w_1 = w_2 = 0$

$\Rightarrow c_1u_1 + c_2u_2 + ... + c_mu_m = d_1v_1 + d_2v_2 + ... + d_nv_n = 0$

$\Rightarrow c_1 = c_2 = ... = c_m = 0$ and $d_1 = d_2 = ... = d_n = 0$

$\Rightarrow \{u_1,u_2,...,u_m, v_1,v_2,...,v_n\}$ is linear independent

$\Rightarrow \{u_1,u_2,...,u_m, v_1,v_2,...,v_n\}$ is the base of W_1+W_2

$\Rightarrow \dim(W_1+W_2)=m+n=\dim(W_1)+\dim(W_2)$

step2 : prove $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)$ $\Rightarrow$ W_1+W_2 is a direct sum

let $S=\{s_1,s_2,...s_k\}$ is the base of $W_1 \cap W_2$

let $U=\{u_1,u_2,...,u_m,s_1,s_2,...s_k\}$ is the base of W_1

let $V=\{v_1,v_2,...,v_n,s_1,s_2,...s_k\}$ is the base of W_2

$S \cap U = \phi$ and $S \cap V = \phi$

$\Rightarrow \dim(W_1 \cap W_2)=k$

$\Rightarrow \dim(W_1)=m+k$

$\Rightarrow \dim(W_2)=n+k$

$\Rightarrow W_1+W_2 = \text{span}\{u_1,u_2,...,u_m,s_1,s_2,...s_k, v_1,v_2,...,v_n\}$

$\because v_1, v_2,..., v_n \notin W_1$

$\therefore v_1, v_2,..., v_n$ can't be represented by $u_1,u_2,...,u_m,s_1,s_2,...s_k$

$\because u_1, u_2,..., u_n \notin W_2$

$\therefore u_1, u_2,..., u_n$ can't be represented by $v_1,v_2,...,v_m,s_1,s_2,...s_k$

$\Rightarrow \{u_1,u_2,...,u_m,s_1,s_2,...s_k, v_1,v_2,...,v_n\}$ is linear independent

$\Rightarrow \{u_1,u_2,...,u_m,s_1,s_2,...s_k, v_1,v_2,...,v_n\}$ is the base of W_1+W_2

$\Rightarrow \dim(W_1+W_2) = m + k + n$

$\because \dim(W_1+W_2) = \dim(W_1)+\dim(W_2)$

$\Rightarrow m + k + n = m + n + 2k$

$\Rightarrow k = 0$

$\Rightarrow \dim(W_1 \cap W_2)=k=0$

$\Rightarrow W_1 \cap W_2 = \Phi$

$\Rightarrow W_1+W_2$ is a direct sum

by step1 and step2 : $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)$ $\Leftrightarrow$ W_1+W_2 is a direct sum

#2

let $W=\{w_1,w_2,\dots,w_n\}$

$$T(v_j)=\sum_{i=1}^{n}a_{ij}w_i=w_j \quad , \quad j=1,\dots,n$$

$$T(v_j)=a_{1j}w_1+a_{2j}w_2+\dots+a_{jj}w_j+\dots+a_{nj}w_n=w_j \quad , \quad j=1,\dots,n$$

$a_{ij}=0 \quad , \quad i\neq j$

$a_{ij}=1 \quad , \quad i=j$

$$\Rightarrow [T]_W=\begin{pmatrix}1&0&\cdots&0\\0&1&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\dots&0&1\end{pmatrix}=I_{n\times n}$$

$\because I_{n\times n}$ is an invertible matrix

$\therefore T$ is an invertible linear transform

#3

let $S=\{\begin{pmatrix}1&0\\0&0\end{pmatrix} , \begin{pmatrix}0&1\\0&0\end{pmatrix} , \begin{pmatrix}0&0\\1&0\end{pmatrix} , \begin{pmatrix}0&0\\0&1\end{pmatrix}\}$

$$T(\begin{pmatrix}1&0\\0&0\end{pmatrix})=\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}$$

$$=1\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}+1\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$T(\begin{pmatrix}0&1\\0&0\end{pmatrix})=\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix}$$

$$=0\begin{pmatrix}1&0\\0&0\end{pmatrix}+1\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\begin{pmatrix}0&0\\1&0\end{pmatrix}+1\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$T(\begin{pmatrix}0&0\\1&0\end{pmatrix})=\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}$$

$$=1\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}+1\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$T(\begin{pmatrix}0&0\\0&1\end{pmatrix})=\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix}$$

$$=0\begin{pmatrix}1&0\\0&0\end{pmatrix}+1\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\begin{pmatrix}0&0\\1&0\end{pmatrix}+1\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\Rightarrow [T]_S=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}$$

#4

$$A=\begin{pmatrix}0 & a_{12} & \cdots & a_{1n}\\ a_{21} & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & a_{n-1\ n}\\ a_{n1} & \dots & a_{n\ n-1} & 0\end{pmatrix}=BC-CB$$

let B is a diagonal matrix

$$A=\begin{pmatrix}0 & a_{12} & \cdots & a_{1n}\\ a_{21} & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & a_{n-1\ n}\\ a_{n1} & \dots & a_{n\ n-1} & 0\end{pmatrix}=\begin{pmatrix}b_{11} & 0 & \cdots & 0\\ 0 & b_{22} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & b_{nn}\end{pmatrix}\begin{pmatrix}c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \ddots & \vdots\\ \vdots & \ddots & \ddots & c_{n-1\ n}\\ c_{n1} & \dots & c_{n\ n-1} & c_{nn}\end{pmatrix}-\begin{pmatrix}c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \ddots & \vdots\\ \vdots & \ddots & \ddots & c_{n-1\ n}\\ c_{n1} & \dots & c_{n\ n-1} & c_{nn}\end{pmatrix}\begin{pmatrix}b_{11} & 0 & \cdots & 0\\ 0 & b_{22} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & b_{nn}\end{pmatrix}$$

$$\Rightarrow a_{ij}=b_{ii}c_{ij}-c_{ij}b_{jj}$$

$$\Rightarrow c_{ij}=\frac{a_{ij}}{b_{ii}-b_{jj}}$$

$$\Rightarrow A=\begin{pmatrix}0 & a_{12} & \cdots & a_{1n}\\ a_{21} & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & a_{n-1\ n}\\ a_{n1} & \dots & a_{n\ n-1} & 0\end{pmatrix}$$

$$=\begin{pmatrix}b_{11} & 0 & \cdots & 0\\ 0 & b_{22} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & b_{nn}\end{pmatrix}\begin{pmatrix}0 & \frac{a_{12}}{b_{11}-b_{22}} & \cdots & \frac{a_{1n}}{b_{11}-b_{nn}}\\ \frac{a_{21}}{b_{22}-b_{11}} & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \frac{a_{n-1\ n}}{b_{n-1\ n-1}-b_{nn}}\\ \frac{a_{n1}}{b_{nn}-b_{11}} & \dots & \frac{a_{n\ n-1}}{b_{nn}-b_{n-1\ n-1}} & 0\end{pmatrix}-\begin{pmatrix}0 & \frac{a_{12}}{b_{11}-b_{22}} & \cdots & \frac{a_{1n}}{b_{11}-b_{nn}}\\ \frac{a_{21}}{b_{22}-b_{11}} & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \frac{a_{n-1\ n}}{b_{n-1\ n-1}-b_{nn}}\\ \frac{a_{n1}}{b_{nn}-b_{11}} & \dots & \frac{a_{n\ n-1}}{b_{nn}-b_{n-1\ n-1}} & 0\end{pmatrix}\begin{pmatrix}b_{11} & 0 & \cdots & 0\\ 0 & b_{22} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & b_{nn}\end{pmatrix}$$

$\Rightarrow$ Every diagonal elements of matrix B must be distinct.

$\Rightarrow$ There exist matrices $B=[b_{ij}]$ and $C=[c_{ij}]$ such that $A=BC-CB$

#5

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \Rightarrow A^{-1}=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

$$AB=BA \Rightarrow B=A^{-1}BA$$

Assume $B=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ $a,b,c,d\in\mathbb{R}$

$$=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & a+b\\ c & c+d\end{pmatrix}$$

$$=\begin{pmatrix}a-c & (a+b)-(c+d)\\ c & c+d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$\Rightarrow c=0\ ,\ a=d$$

$$\Rightarrow B=\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}=bA+(a-b)I=p(A)$$

$$\Rightarrow p(A)=bA+(a-b)I$$

$$\Rightarrow p(t)=bt+(a-b)$$

$\Rightarrow$ There exists a polynomial $p(t)=bt+(a-b)$ such that $B=p(A)$

#6

$$Ax = y$$

$$\Rightarrow (I - 2uu^T)x = y$$

$$\Rightarrow x - 2uu^T x = y$$

$$\Rightarrow x - 2u(u^T x) = y$$

$$\Rightarrow x - 2(u^T x)u = y$$

$$\Rightarrow 2(u^T x)u = x - y$$

$\Rightarrow$ u is on the direction of $x - y$

let $u = k(x - y)$

$$\Rightarrow 2(u^T x)k(x - y) = x - y$$

$$\Rightarrow 2(u^T x)k = 1$$

$$\Rightarrow 2[k(x - y)^T x]k = 1$$

$$\Rightarrow 2k^2[x^T x - y^T x] = 1$$

$$\Rightarrow k^2 = \frac{1}{2[x^T x - y^T x]}$$

$$\Rightarrow k = \pm \frac{1}{\sqrt{2[x^T x - y^T x]}}$$

(1) $x = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow x^T x = 1 \quad \text{and} \quad y^T x = \frac{1}{3}$$

$$\Rightarrow k = \pm \frac{1}{\sqrt{2[1 - \frac{1}{3}]}} = \pm \frac{\sqrt{3}}{2}$$

$$\Rightarrow u = k(x - y)$$

$$\Rightarrow u = \pm \frac{\sqrt{3}}{2}[\ \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\ ]$$

$$\Rightarrow u = \pm \frac{\sqrt{3}}{2}[\ \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}\ ] = \pm \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow A = I - 2uu^T$$

$$\Rightarrow A = I - 2(\frac{1}{3})\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

(2)

$\Rightarrow$ u is on the direction of $z - y$

$\Rightarrow$ let $u = c(z - y)$

$$u = c(\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = c\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow c = \pm \frac{1}{\sqrt{2[z^T z - y^T z]}} = \pm \frac{1}{\sqrt{2[14 - (-1)]}} = \pm \frac{1}{\sqrt{30}}$$

$$\Rightarrow u = \pm \frac{1}{\sqrt{30}} \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow |u| = \sqrt{\frac{17}{30}} \neq 1$$

$\Rightarrow$ It is impossible to find a matrix A such that $Az = y$, where $z = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$