Solutions to problem set3

#1.

(a)

$$[A\,|\,I]=\left(\begin{array}{ccccc|ccccc} 0 & a_1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \ddots & \vdots & 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & a_{n-1} & \vdots & \vdots & \ddots & 1 & 0 \\ a_n & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$\Rightarrow R_{n-1\ n}, R_{n-2\ n-1}, \ldots R_{2\ 3}, R_{1\ 2} \Rightarrow \left(\begin{array}{ccccc|ccccc} a_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & a_1 & 0 & \ddots & \vdots & 1 & 0 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-2} & 0 & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_{n-1} & 0 & \cdots & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow R_1^{(a_n^{-1})}, R_2^{(a_1^{-1})}, \ldots, R_{n-1}^{(a_{n-2}^{-1})}, R_n^{(a_{n-1}^{-1})} \Rightarrow \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_n^{-1} \\ 0 & 1 & 0 & \ddots & \vdots & a_1^{-1} & 0 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 & \vdots & \ddots & a_{n-2}^{-1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & a_{n-1}^{-1} & 0 \end{array}\right)$$

$$\Rightarrow A^{-1}=\begin{pmatrix} 0 & 0 & \cdots & 0 & a_n^{-1} \\ a_1^{-1} & 0 & 0 & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & a_{n-2}^{-1} & 0 & 0 \\ 0 & \cdots & 0 & a_{n-1}^{-1} & 0 \end{pmatrix}$$

(b)

$$[B\,|\,I]=\left(\begin{array}{ccccc|ccccc} 1 & 2 & 3 & \cdots & n & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \ddots & \vdots & 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 3 & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 2 & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$\Rightarrow R_{21}^{-1}, R_{32}^{-1}, \ldots, R_{n-1\ n-2}^{-1}, R_{n\ n-1}^{-1} \Rightarrow \left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & \cdots & 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots & 0 & 1 & -1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 1 & \vdots & \vdots & \ddots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$\Rightarrow R_{21}^{-1}, R_{32}^{-1}, \ldots, R_{n-1\ n-2}^{-1}, R_{n\ n-1}^{-1} \Rightarrow \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots & 0 & 1 & -2 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 & \vdots & 0 & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & 1 & 0 & \vdots & \vdots & \ddots & 1 & -2 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$\Rightarrow A^{-1}=\begin{pmatrix} 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & 1 & -2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

#2.

$\because A+B$ is invertible

$\therefore (A+B)^{-1}(A+B)=I$

$\Rightarrow (A+B)^{-1}A+(A+B)^{-1}B=I$

$\Rightarrow (A+B)^{-1}A=I-(A+B)^{-1}B$

Consider

$[A-A(A+B)^{-1}A]-[B-B(A+B)^{-1}B]$

$=[A-A(A+B)^{-1}A]-B[I-(A+B)^{-1}B]$

$=[A-A(A+B)^{-1}A]-B[(A+B)^{-1}A]$

$=A-(A+B)(A+B)^{-1}A$

$=A-IA$

$=0$

$\Rightarrow A-A(A+B)^{-1}A=B-B(A+B)^{-1}B$

#3 A,B is invertible and $n\times n$

(a)

$$\left(\begin{array}{cc|cc} A & C & I_n & 0 \\ 0 & B & 0 & I_n \end{array}\right)=\left(\begin{array}{cc|cc} A & 0 & I_n & -CB^{-1} \\ 0 & B & 0 & I_n \end{array}\right)=\left(\begin{array}{cc|cc} I_n & 0 & A^{-1} & -A^{-1}CB^{-1} \\ 0 & I_n & 0 & B^{-1} \end{array}\right)$$

(b)

$$\left(\begin{array}{cc|cc} 0 & A & I_n & 0 \\ B & 0 & 0 & I_n \end{array}\right)=\left(\begin{array}{cc|cc} B & 0 & 0 & I_n \\ 0 & A & I_n & 0 \end{array}\right)=\left(\begin{array}{cc|cc} I_n & 0 & 0 & B^{-1} \\ 0 & I_n & A^{-1} & 0 \end{array}\right)$$

(c)

$$\left(\begin{array}{cc|cc} A & I_n & I_n & 0 \\ I_n & 0 & 0 & I_n \end{array}\right)=\left(\begin{array}{cc|cc} 0 & I_n & I_n & -A \\ I_n & 0 & 0 & I_n \end{array}\right)=\left(\begin{array}{cc|cc} I_n & 0 & 0 & I_n \\ 0 & I_n & I_n & -A \end{array}\right)$$

#4

(1)

$$[A\,|\,I]=\left(\begin{array}{ccccc|cccc}1&0&2&1&4&1&0&0&0\\1&1&3&1&6&0&1&0&0\\1&1&4&0&3&0&0&1&0\\1&1&5&0&3&0&0&0&1\end{array}\right)$$

$$\Rightarrow R_{12}^{-1},R_{13}^{-1},R_{14}^{-1}\Rightarrow\left(\begin{array}{ccccc|cccc}1&0&2&1&4&1&0&0&0\\0&1&1&0&2&-1&1&0&0\\0&1&2&-1&-1&-1&0&1&0\\0&1&3&-1&-1&-1&0&0&1\end{array}\right)$$

$$\Rightarrow R_{23}^{-1},R_{24}^{-1}\Rightarrow\left(\begin{array}{ccccc|cccc}1&0&2&1&4&1&0&0&0\\0&1&1&0&2&-1&1&0&0\\0&0&1&-1&-3&0&-1&1&0\\0&0&2&-1&-3&0&-1&0&1\end{array}\right)$$

$$\Rightarrow R_{34}^{-2}\Rightarrow\left(\begin{array}{ccccc|cccc}1&0&2&1&4&1&0&0&0\\0&1&1&0&2&-1&1&0&0\\0&0&1&-1&-3&0&-1&1&0\\0&0&0&1&3&0&1&-2&1\end{array}\right)$$

$$\Rightarrow R_{43}^{1},R_{41}^{-1}\Rightarrow\left(\begin{array}{ccccc|cccc}1&0&2&0&1&1&-1&2&-1\\0&1&1&0&2&-1&1&0&0\\0&0&1&0&0&0&0&-1&1\\0&0&0&1&3&0&1&-2&1\end{array}\right)$$

$$\Rightarrow R_{31}^{-2},R_{32}^{-1}\Rightarrow\left(\begin{array}{ccccc|cccc}1&0&0&0&1&1&-1&4&-3\\0&1&0&0&2&-1&1&1&-1\\0&0&1&0&0&0&0&-1&1\\0&0&0&1&3&0&1&-2&1\end{array}\right)=[R\,|\,E]$$

$$\Rightarrow \text{base of colum space } A=\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\end{pmatrix},\begin{pmatrix}2\\3\\4\\5\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\}$$

$$\Rightarrow \text{base of row space } A=\{\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\3\end{pmatrix}\}$$

(2)

let $X \in N(A)$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$

$$\Rightarrow AX = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 & 0 & 0 & 0 & x_5 \\ 0 & x_2 & 0 & 0 & 2x_5 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 3x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_5 \\ -2x_5 \\ 0 \\ -3x_5 \\ x_5 \end{pmatrix} = x_5 \begin{pmatrix} -1 \\ -2 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{base of null space } A = \{ \begin{pmatrix} -1 \\ -2 \\ 0 \\ -3 \\ 1 \end{pmatrix} \}$$

(3)

let $X \in N(A^T)$ $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$\Rightarrow A^T X = 0$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ 1 & 1 & 3 & 1 & 6 \\ 1 & 1 & 4 & 0 & 3 \\ 1 & 1 & 5 & 0 & 3 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ x_2 + x_3 + x_4 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 \\ x_1 + x_2 \\ 4x_1 + 6x_2 + 3x_3 + 3x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ only $x_1 = x_2 = x_3 = x_4 = 0$ satisfied

$\Rightarrow$ the base for left nullspace of $A = \Phi$

(4)

$$R=\begin{pmatrix}1&0&0&0&1\\0&1&0&0&2\\0&0&1&0&0\\0&0&0&1&3\end{pmatrix}$$

$\Rightarrow$ number of pivot for rows of $A=$ number of pivot for colums of $A=4$

$\Rightarrow \dim(C(A))=\dim(R(A))=\mathrm{rank}(A)=4$

#5

(a)

$$\Rightarrow\left(\begin{array}{cc|cc}I&A&I&0\\B&I&0&I\end{array}\right)\Rightarrow C_{21}^{(-B)}\Rightarrow\left(\begin{array}{cc|cc}I-AB&A&I&0\\0&I&-B&I\end{array}\right)\Rightarrow R_{21}^{(-A)}\Rightarrow\left(\begin{array}{cc|cc}I-AB&0&I+AB&-A\\0&I&-B&I\end{array}\right)$$

$$\Rightarrow\left(\begin{array}{cc|cc}I&A&I&0\\B&I&0&I\end{array}\right)\Rightarrow C_{12}^{(-A)}\Rightarrow\left(\begin{array}{cc|cc}I&0&I&-A\\B&I-BA&0&I\end{array}\right)\Rightarrow R_{12}^{(-B)}\Rightarrow\left(\begin{array}{cc|cc}I&0&I&-A\\0&I-BA&-B&I+BA\end{array}\right)$$

$$\Rightarrow \mathrm{rank}(\begin{pmatrix}I&A\\B&I\end{pmatrix})=\mathrm{rank}(\begin{pmatrix}I-AB&0\\0&I\end{pmatrix})=\mathrm{rank}(\begin{pmatrix}I&0\\0&I-BA\end{pmatrix})$$

$$\because \mathrm{rank}(\begin{pmatrix}I-AB&0\\0&I\end{pmatrix})=\mathrm{rank}(I-AB)+\mathrm{rank}(I)=\mathrm{rank}(I-AB)+n$$

$$\mathrm{rank}(\begin{pmatrix}I&0\\0&I-BA\end{pmatrix})=\mathrm{rank}(I-BA)+\mathrm{rank}(I)=\mathrm{rank}(I-BA)+n$$

$$\therefore \mathrm{rank}(I-AB)=\mathrm{rank}(I-BA)$$

(b)

$$\left(\begin{array}{cc|cc}A&0&I&0\\0&B&0&I\end{array}\right)\Rightarrow C_{21}^{(1)}\Rightarrow\left(\begin{array}{cc|cc}A&0&I&0\\B&B&I&I\end{array}\right)\Rightarrow R_{21}^{(1)}\Rightarrow\left(\begin{array}{cc|cc}A+B&B&2I&I\\B&B&I&I\end{array}\right)$$

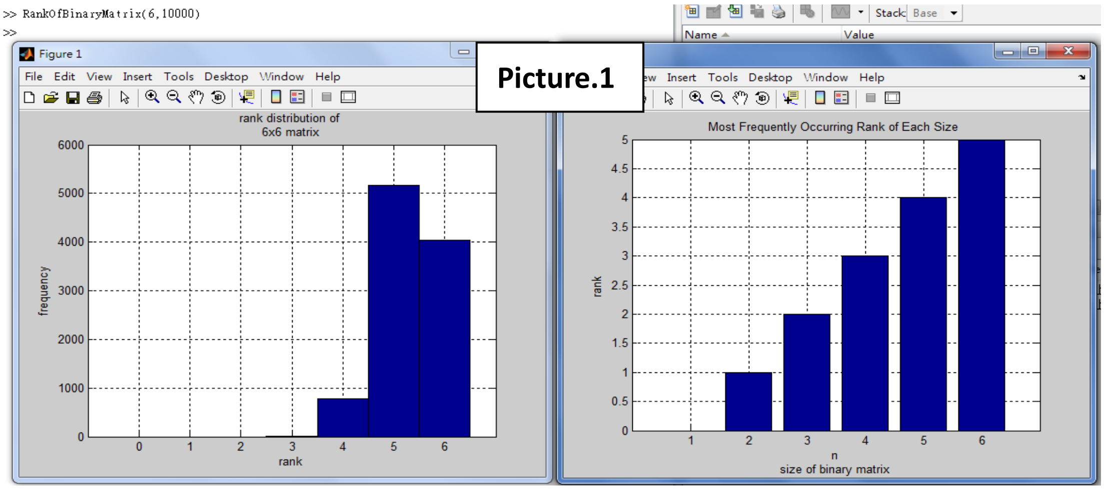
$$\Rightarrow \mathrm{rank}(\begin{pmatrix}A&0\\0&B\end{pmatrix})=\mathrm{rank}(\begin{pmatrix}A+B&B\\B&B\end{pmatrix})$$

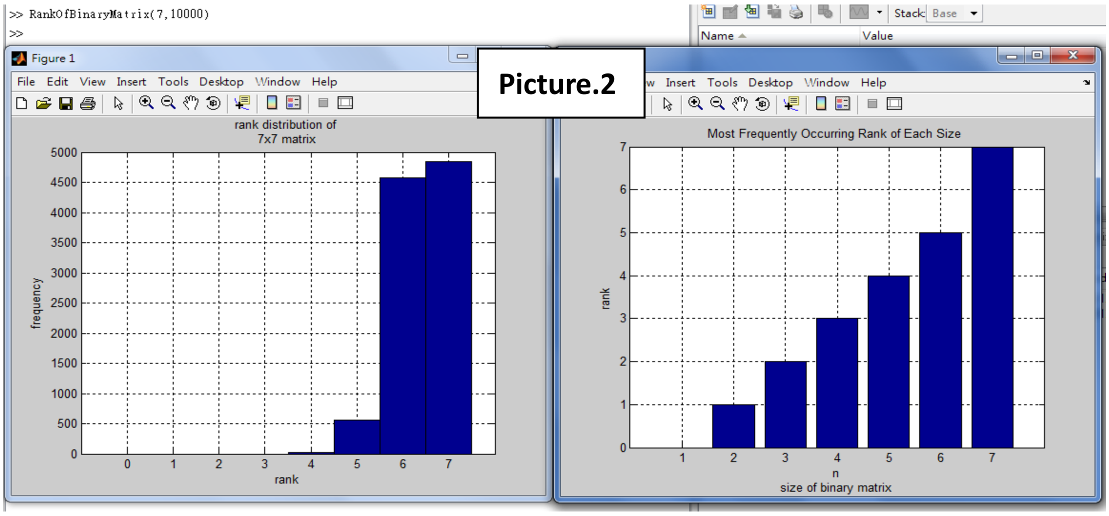
$$\Rightarrow \mathrm{rank}(A)+\mathrm{rank}(B)=\mathrm{rank}(\begin{pmatrix}A+B&B\\B&B\end{pmatrix})$$

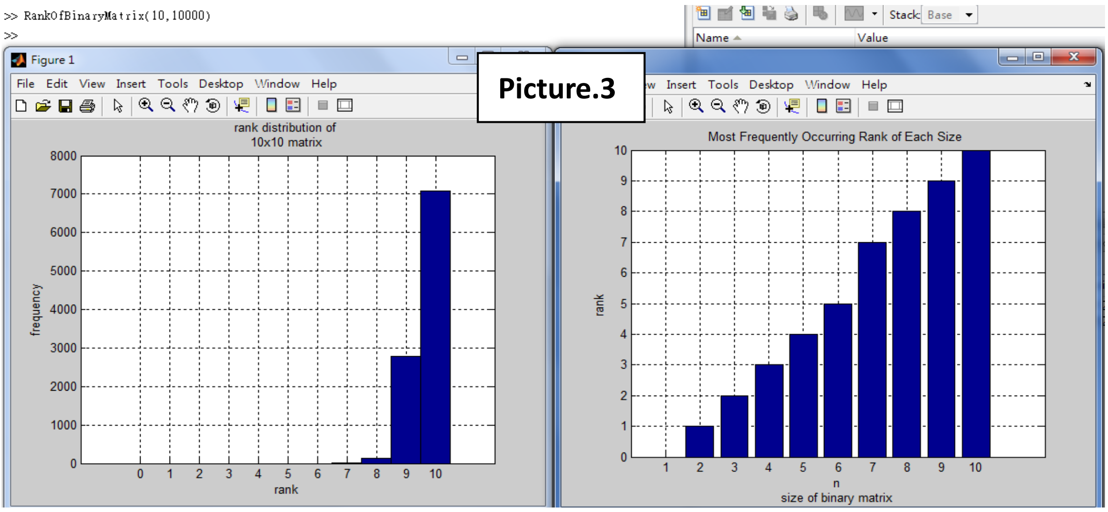
obviously $\mathrm{rank}(\begin{pmatrix}A+B&B\\B&B\end{pmatrix})\geq \mathrm{rank}(A+B)$

$$\Rightarrow \mathrm{rank}(\begin{pmatrix}A&0\\0&B\end{pmatrix})\geq \mathrm{rank}(A+B)$$

#6







(a) As we can see from the figures above, when matrix size is n, most of the ranks appear at

n-1 and n no matter how n increases.

(b) $n \leq 6$: The most frequently occurring rank is n-1.

$n > 6$: The most frequently occurring rank is n.

Refer to HW1, we can see that as n increases and exceeds 6, the fraction of singular matrices decreases and drops below 0.5. That is, about 50% of the matrices become invertible (full rank) when n>6. This can be observed easily from figures above (Picture.1 to Picture.3). Therefore, the most frequently occurring rank with respect to n has the tendencies described above.