Solutions to problem set 4

#1

step1:

let $c_1v_1 + c_2v_2 + c_3w_1 + c_4w_2 + c_5w_3 = 0$

$$\Rightarrow [v_1 \quad v_2 \quad w_1 \quad w_2 \quad w_3]\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 1 & -1 & 3 \\ 2 & 5 & 2 & 0 & 1 \\ -7 & -6 & -1 & 1 & 1 \\ 3 & -5 & -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0$$

$\Rightarrow c_1 = 0 \quad c_2 = c_5 \quad c_3 = -3c_4 \quad c_4 = 2c_5$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ c_5 \\ -3c_5 \\ 2c_5 \\ c_5 \end{pmatrix} = c_5\begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

step2:

$$\text{rank}[v_1 \quad v_2 \quad w_1 \quad w_2 \quad w_3] = \text{rank}(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}) = 4$$

$\Rightarrow \dim(V+W) = 4$

$\Rightarrow \dim(V \cap W) = \dim(V) + \dim(W) - \dim(V+W)$

$\Rightarrow \dim(V \cap W) = 2 + 3 - 4 = 1$

step3:

$c_1v_1 + c_2v_2 + c_3w_1 + c_4w_2 + c_5w_3 = 0$

$\Rightarrow 0v_1 + 1v_2 + (-3)w_1 + 2w_2 + 1w_3 = 0$

$\Rightarrow v_2 = 3w_1 - 2w_2 - 1w_3$

$\Rightarrow v_2 \in V$ and $(3w_1 - 2w_2 - 1w_3) \in W$

$\Rightarrow v_2 = 3w_1 - 2w_2 - 1w_3 \in V \cap W$

$\Rightarrow v_2 = \begin{pmatrix} 2 \\ 5 \\ -6 \\ -5 \end{pmatrix}$ is the base of $V \cap W$

$\because v_2$ can be represented by w_1, w_2, w_3

$\therefore \{v_1 \ , \ w_1 \ , \ w_2 \ , \ w_3\}$ is the base of $V + W$

#2

(a)

by theorem : $\text{rank}A + \text{rank}B - n \le \text{rank}(AB) \le \min(\text{rank}A, \text{rank}B)$

$\Rightarrow 4 + \text{rank}B - 7 \le \text{rank}(AB) \le \min(4, \text{rank}B)$

if $\text{rank}B = 6$ then $3 \le \text{rank}(AB) \le 4$

if $\text{rank}B = 5$ then $2 \le \text{rank}(AB) \le 4$

if $\text{rank}B = 4$ then $1 \le \text{rank}(AB) \le 4$

if $\text{rank}B = 3$ then $0 \le \text{rank}(AB) \le 3$

$\because \text{rank}(AB) \ge 0$

if $\text{rank}B = 2$ then $0 \le \text{rank}(AB) \le 2$

if $\text{rank}B = 1$ then $0 \le \text{rank}(AB) \le 1$

if $\text{rank}B = 0$ then $0 \le \text{rank}(AB) \le 0$

$\Rightarrow 0 \le \text{rank}(AB) \le 4$

(b)

step1

$\because \text{rank}(B) \ge \text{rank}(AB)$

$\therefore \text{rank}(B) \ge 2$

step2

$\because$ the size of B is 7×6

$\therefore \text{rank}B \le \min(7,6) = 6$

by step1 and step2

$\Rightarrow 2 \le \text{rank}B \le 6$

(c)

by theorem : $\text{rank}(A) + \text{rank}(B) - n \le \text{rank}(AB) \le \min(\text{rank}A, \text{rank}B)$

$\Rightarrow 3 + 5 - 7 \le \text{rank}(AB) \le \min(3,5)$

$\Rightarrow 1 \le \text{rank}(AB) \le 3$

$\Rightarrow$ It is impossible that $AB = 0$

#3

let W,U,V are the three 6 – dimensional subspace in $\mathbb{R}^8$

$\Rightarrow \dim W = \dim U = \dim V = 6$

$\dim(W+U) = \dim W + \dim U - \dim(W \cap U)$

$\Rightarrow \dim(W \cap U) = \dim W + \dim U - \dim(W+U)$

$\Rightarrow \dim(W \cap U) = 6 + 6 - \dim(W+U)$

$\because \dim(W+U) \le \dim \mathbb{R}^8 = 8$

$\therefore \dim(W \cap U) \ge 6 + 6 - 8$

$\Rightarrow \dim(W \cap U) \ge 4$

$\dim(W \cap U \cap V) = \dim(W \cap U) + \dim V - \dim((W \cap U) + V)$

$\dim(W \cap U \cap V) = \dim(W \cap U) + 6 - \dim((W \cap U) + V)$

$\because \dim((W \cap U) + V) \le \dim \mathbb{R}^8 = 8$ and $\dim(W \cap U) \ge 4$

$\therefore \dim(W \cap U \cap V) \ge 4 + 6 - 8$

$\Rightarrow \dim(W \cap U \cap V) \ge 2$

$\Rightarrow$ the intersection of three 6 – dimemsional subspace in $\mathbb{R}^8$ is not the single point {0}

#4

let $A = [A_1 \ \ A_2 \ \ \dots \ \ A_n]$

$A_1, A_2, \dots, A_n$ is $m \times 1$ matrix

$A_1, A_2, \dots, A_n \in C(A)$

$$EA = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} [A_1 \ \ A_2 \ \ \dots \ \ A_n] = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E_1A_1 & E_1A_2 & \dots & E_1A_n \\ E_2A_1 & E_2A_2 & \dots & E_2A_n \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$\Rightarrow E_2A_1 = E_2A_2 = \cdots = E_2A_n = 0$

$\Rightarrow A_1, A_2, \dots, A_n \in N(E_2)$

$\because \operatorname{rank}(A) = r$

$\therefore \dim(C(A)) = r$

let $A_1, A_2, \dots, A_r$ is the base of $C(A)$

let $\forall A_x = c_1A_1 + c_2A_2 + \cdots + c_rA_r \in C(A)$

$E_2A_x = E_2(c_1A_1 + c_2A_2 + \cdots + c_rA_r)$

$\Rightarrow E_2A_x = c_1E_2A_1 + c_2E_2A_2 + \cdots + c_rE_2A_r = 0 \quad (\because A_1, A_2, \dots, A_r \in N(E_2))$

$\Rightarrow A_x \in N(E_2)$

$\Rightarrow C(A) \subset N(E_2)$

by rank $-$ nullity theorem : $\operatorname{rank}(E_2) + \operatorname{nullity}(E_2) = m$

$\Rightarrow \operatorname{nullity}(E_2) = m - \operatorname{rank}(E_2) = m - (m - r) = r$

$\Rightarrow \operatorname{nullity}(E_2) = \dim(N(E_2)) = r$

$\dim(C(A)) = \dim(N(E_2))$ and $C(A) \subset N(E_2)$

$\Rightarrow C(A) = N(E_2)$

#5

let $U = [u_1\ u_2 \cdots\ u_n]$

$v = [v_1\ v_2 \cdots\ v_n]$

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

$Au_j = c_{1j}u_1 + c_{2j}u_2 + \dots + c_{nj}u_n$

$\Rightarrow AU = UC \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$

$Bu_j = c_{1j}v_1 + c_{2j}v_2 + \dots + c_{nj}v_n$

$\Rightarrow BV = VC$

$\because U, V$ are invertible

$\therefore C = V^{-1}BV$ apply to (1)

$\Rightarrow AU = U(V^{-1}BV)$

$\Rightarrow A = U(V^{-1}BV)U^{-1} = (UV^{-1})B(VU^{-1}) = SBS^{-1}$

$S = UV^{-1}$ and $S^{-1} = VU^{-1}$

$\because U, V$ are invertible

$\therefore S^{-1} = VU^{-1}$ exists

#6

(1)

U and W are invariant subspace under linear transformation A

$\Rightarrow A(U) \subseteq U$ and $A(W) \subseteq W$

for any $u_t \in U$, $w_t \in W$

$\Rightarrow A(u_t) \in U$ and $A(w_t) \in W$

consider $u_t + w_t \in U + W$

$\Rightarrow A(u_t + w_t) = A(u_t) + A(w_t) \in U + W$

$\Rightarrow A(U + W) \subseteq U + W$

$\Rightarrow U + W$ is an invarient subspace under A

(2)

for any $x_t \in U \cap W$

$\Rightarrow x_t \in U$ and $x_t \in W$

$\Rightarrow A(x_t) \in U$ and $A(x_t) \in W$

($\because U, W$ are invarient subspaces under A)

$\Rightarrow A(x_t) \in U \cap W$

$\Rightarrow A(U \cap W) \subseteq U \cap W$

$\Rightarrow U \cap W$ is an invarient subspace under A