Solutions to problem set 5

#1

(a)

Case1 : $\text{rank}(A) = n - 1$

if A is singular then $\text{rank}(A) \le n-1$

$\because$ A has a nonsingular $(n-1)\times(n-1)$ submatrix

$\therefore$ $\text{adj}A \ne 0_{n\times n}$

$\Rightarrow \text{rank}(\text{adj}A) \ge 1 \cdots\cdots\cdots\cdots (1)$

$\det(A)\times I_{n\times n} = \text{adj}A \times A$

$\because \det(A) = 0$

$\therefore \text{adj}A \times A = 0$

$\Rightarrow C(A) \subseteq N(\text{adj}A)$

$\Rightarrow \dim(C(A)) \le \text{Nullity}(\text{adj}A)$

$\Rightarrow n - 1 \le n - \text{rank}(\text{adj}A)$

$\Rightarrow \text{rank}(\text{adj}A) \le 1 \cdots\cdots\cdots\cdots (2)$

By (1) and (2)

if $\text{rank}(A) = n - 1$ then $\text{rank}(\text{adj}A) = 1$

Case2 : $\text{rank}A < n - 1$

all $(n-1)\times(n-1)$ submatrice of A are singular

$\text{adj}A = 0_{n\times n}$

$\Rightarrow$ if $\text{rank}A < n - 1$ then $\text{rank}(\text{adj}A) = 0$

By Case1 and Case2

$\text{rank}(\text{adj}A) = 0$ or $\text{rank}(\text{adj}A) = 1$

(b)

$\text{rank}(\text{adj}A) = 0$ or $\text{rank}(\text{adj}A) = 1$ and $n \ge 3$

$\therefore n - 1 \ge 2 \Rightarrow \text{rank}(\text{adj}A) < 2 \le n - 1$

$\Rightarrow \text{rank}(\text{adj}A) < n - 1$

By result in (a) : if $\text{rank}A < n - 1$ then $\text{rank}(\text{adj}A) = 0$

$\because \text{rank}(\text{adj}A) < n - 1$

$\therefore \text{rank}(\text{adj}(\text{adj}A)) = 0 \Rightarrow \text{adj}(\text{adj}A) = [0]_{n\times n}$

(c)

$SAS^{-1} \times \text{adj}(SAS^{-1}) = \det(SAS^{-1}) \times I_{n\times n}$

$\Rightarrow SAS^{-1} \times \text{adj}(SAS^{-1}) = \det(A) \times I_{n\times n}$

$\Rightarrow \text{adj}(SAS^{-1}) = \det(A)(SAS^{-1})^{-1}$

$\Rightarrow \text{adj}(SAS^{-1}) = \det(A)(SA^{-1}S^{-1})$

$\Rightarrow \text{adj}(SAS^{-1}) = S(\det(A)A^{-1})S^{-1}$

$\Rightarrow \text{adj}(SAS^{-1}) = S(\text{adj}A)S^{-1}$

#2

(a)

let $u = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$$A = \begin{pmatrix}a & 1 & 1 & 1\\1 & a & 1 & 1\\1 & 1 & a & 1\\1 & 1 & 1 & a\end{pmatrix} = (a-1)I_4 + uu^T$$

$Au = (a-1)I_4u + uu^Tu = (a-1)u + 4u = (a+3)u$

$\Rightarrow \lambda_1 = a + 3$

$\Rightarrow V_1 = u = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$\because$ A is a symmetric matrix

$\therefore$ the eigenvectors of A are orthogonal to each other

let V is another eigenvector of A

$AV = (a-1)I_4V + uu^TV$

$\because$ V is orthogonal to u

$\therefore u^TV_2 = 0$

$\Rightarrow AV = (a-1)I_4V$ and $uu^TV = 0$

$\Rightarrow \lambda_2 = \lambda_3 = \lambda_4 = a - 1$ and $V_2, V_3, V_4 \in N(uu^T)$

$$uu^TV = 0 \Rightarrow \begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow V = \begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = \begin{pmatrix}-v_2 - v_3 - v_4\\v_2\\v_3\\v_4\end{pmatrix} = v_2\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + v_3\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + v_4\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$$

the eigenvalues of $A = a+3, a-1, a-1, a-1$

the eigenvectors of $A = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$

(b)

$$B = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}$$

$$\because \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} + \begin{pmatrix} c \\ d \\ a \\ b \end{pmatrix} + \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ a+b+c+d \\ a+b+c+d \\ a+b+c+d \end{pmatrix}$$

$\therefore$ B has an eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$BV_1 = \lambda_1 V_1 \Rightarrow \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+b+c+d \\ a+b+c+d \\ a+b+c+d \\ a+b+c+d \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{pmatrix} \Rightarrow \lambda_1 = a+b+c+d$$

$\because$ B is a symmetric matrix

$\therefore$ the eigenvectors of B are orthogonal to each other

$\Rightarrow$ other eigenvectors of B are $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$$V_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \lambda_2 = a+b-c-d$$

$$V_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \lambda_3 = a-b-c+d$$

$$V_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \lambda_4 = a-b+c-d$$

the eigenvalues of $B = a+b+c+d$, $a+b-c-d$, $a-b-c+d$, $a-b+c-d$

the eigenvectors of $B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} , \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} , \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

#3

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

$$(A - \lambda I_{(m+n)})V = 0$$

$$\Rightarrow (A - \lambda I_{(m+n)}) = \begin{pmatrix} B - \lambda I_m & 0 \\ 0 & C - \lambda I_n \end{pmatrix}$$

$$\det(A - \lambda I_{(m+n)}) = \det(B - \lambda I_m)\det(C - \lambda I_n) = 0$$

Case1 : for λ satisfies $\det(B - \lambda I_m) = 0$ and $\det(C - \lambda I_n) \neq 0$

$\Rightarrow$ eigenvalue of A = eigenvalue of B

Case2 : for λ satisfies $\det(B - \lambda I_m) \neq 0$ and $\det(C - \lambda I_n) = 0$

$\Rightarrow$ eigenvalue of A = eigenvalue of C

Case3 : for λ satisfies $\det(B - \lambda I_m) = 0$ and $\det(C - \lambda I_n) = 0$

$\Rightarrow$ eigenvalue of A = {eigenvalue of B , eigenvalue of C}

let eigenvector of A is eig_v(A) $\quad$ eig_v(A) $\in \mathbb{R}^{m+n}$

let eigenvector of B is eig_v(B) $\quad$ eig_v(B) $\in \mathbb{R}^{m}$

let eigenvector of C is eig_v(C) $\quad$ eig_v(C) $\in \mathbb{R}^{n}$

in case1:eig_v(A) can be in the form of $\begin{pmatrix} \text{eig\_v(B)} \\ 0 \end{pmatrix}$

in case2 : eig_v(A) can be in the form of $\begin{pmatrix} 0 \\ \text{eig\_v(C)} \end{pmatrix}$

in case3 : eig_v(A) can be in the form of $\begin{pmatrix} \text{eig\_v(B)} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \text{eig\_v(C)} \end{pmatrix}$

#4

(a)

$$A = I + uu^T$$

$$Au = Iu + uu^Tu = u + 5u = 6u$$

$$\Rightarrow \lambda_1 = 6 \quad \text{and} \quad V_1 = u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

let V is another eigenvetors of A

$$AV = IV + uu^TV = V + uu^TV$$

$$\because A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \text{ is a symmetric matrix}$$

$\therefore$ the eigenvectors of A are orthogonal to each other

$\Rightarrow u^T V = 0$

$\Rightarrow AV = V$

$\Rightarrow \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1$

$uu^T V = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} -v_2 - v_3 - v_4 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + v_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

the eigenvalues of $A = 6, 1, 1, 1, 1$

$$\text{the eigenvectors of } A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(b)

$AX = \lambda X$

$$A \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}^{-1}$$

$\Rightarrow A = S \Lambda S^{-1}$

$$\Rightarrow \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

(c)

$\det(A) = \det(\Lambda) = 6$

(d)

$tr(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 = 1 + 1 + 1 + 1 + 6^2 = 40$

#5

(a)

$(q(A) - q(\lambda)I) = (a_0 - q(\lambda))I + a_1 A + \cdots + a_m A^m$

$(q(A) - q(\lambda)I)X = [(a_0 - q(\lambda))I + a_1 A + \cdots + a_m A^m]X$

$= \{[a_0 - (a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_m\lambda^m)]I + a_1 A + \cdots + a_m A^m\}X$

$= \{-(a_1\lambda + a_2\lambda^2 + \cdots + a_m\lambda^m)I + a_1 A + \cdots + a_m A^m\}X$

$= \{a_1[A - \lambda I] + a_2[A^2 - \lambda^2 I] + \cdots + a_m[A^m - \lambda^m I]\} = 0$

$\Rightarrow (q(A) - q(\lambda)I)X = 0$

$\Rightarrow q(\lambda)$ is eigenvalue of $q(A)$, x is eigenvector of $q(A)$

(b)

A is diagonalizable $\Rightarrow$ let $S\Lambda S^{-1} = A$

$q(A) = a_0 I + a_1 A + a_2 A^2 \cdots + a_m A^m$

$\Rightarrow q(A) = a_0 I + a_1(S\Lambda S^{-1}) + a_2(S\Lambda S^{-1})^2 + \cdots + a_m(S\Lambda S^{-1})^m$

$\Rightarrow q(A) = a_0 I + a_1(S\Lambda S^{-1}) + a_2(S\Lambda^2 S^{-1}) + \cdots + a_m(S\Lambda^m S^{-1})$

(c)

$q(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$

$\Rightarrow$ A is similar to B

$\Rightarrow A = MBM^{-1}$

$\Rightarrow q(A) = a_0 I + a_1 MBM^{-1} + a_2(MBM^{-1})^2 + \cdots + a_m(MBM^{-1})^m$

$\Rightarrow q(A) = a_0 I + a_1 MBM^{-1} + a_2 MB^2 M^{-1} + \cdots + a_m MB^m M^{-1}$

$\Rightarrow q(A) = Ma_0 IM^{-1} + a_1 MBM^{-1} + a_2 MB^2 M^{-1} + \cdots + a_m MB^m M^{-1}$

$\Rightarrow q(A) = M(a_0 I + a_1 B + a_2 B^2 + \cdots + a_m B^m)M^{-1}$

$\Rightarrow q(A)$ is similar to $q(B)$

#6

let A is a $n \times n$ matrix

let $p_A(t) = \det(A - tI_n) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$

$\because$ A is diagonalizable

$A = S\Lambda S^{-1}$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$p_A(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$

$\Rightarrow p_A(A) = (A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$

$\Rightarrow p_A(S\Lambda S^{-1}) = (S\Lambda S^{-1} - \lambda_1 I)(S\Lambda S^{-1} - \lambda_2 I)\cdots(S\Lambda S^{-1} - \lambda_n I)$

$\Rightarrow p_A(S\Lambda S^{-1}) = (S\Lambda S^{-1} - S\lambda_1 I S^{-1})(S\Lambda S^{-1} - S\lambda_2 I S^{-1})\cdots(S\Lambda S^{-1} - S\lambda_n I S^{-1})$

$\Rightarrow p_A(S\Lambda S^{-1}) = S(\Lambda - \lambda_1 I)S^{-1}S(\Lambda - \lambda_2 I)S^{-1}\cdots S(\Lambda - \lambda_n I)S^{-1}$

$\Rightarrow p_A(S\Lambda S^{-1}) = S(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)S^{-1}$

$$\Rightarrow p_A(S\Lambda S^{-1}) = S\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda_1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n - \lambda_1 \end{pmatrix}\begin{pmatrix} \lambda_1 - \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n - \lambda_2 \end{pmatrix} \times \cdots \times \begin{pmatrix} \lambda_1 - \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda_n & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} S^{-1}$$

let $E_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda_1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n - \lambda_1 \end{pmatrix}$, $E_2 = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n - \lambda_2 \end{pmatrix}, \cdots, E_n = \begin{pmatrix} \lambda_1 - \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda_n & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$

$\Rightarrow E_1 \times E_2$: colum1, colum2 $= [0]_{n\times 1}$ row1, row2 $= [0]_{1\times n}$

$\Rightarrow E_1 \times E_2 \times E_3$: colum1, colum2, column3 $= [0]_{n\times 1}$ row1, row2, row3 $= [0]_{1\times n}$

$\Rightarrow E_1 \times E_2 \times \cdots \times E_3$: colum1, colum2, $\cdots$, column n $= [0]_{n\times 1}$ row1, row2, $\cdots$, row n $= [0]_{1\times n}$

$\Rightarrow E_1 \times E_2 \times \cdots \times E_3 = 0 \Rightarrow p_A = 0$