Solutions to problem set 6

#1

(a)

the eigenvalues of $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = 1,0$

$\lambda = 1 \Rightarrow$ eigenvector $X = \begin{pmatrix}1\\ 0\end{pmatrix}$

$\lambda = 0 \Rightarrow$ eigenvector $X = \begin{pmatrix}1\\ -1\end{pmatrix}$

$AX = \lambda X$

$$\Rightarrow \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}^{-1}$$

$\Rightarrow \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ are similar

(b)

$$\because \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

$\Rightarrow \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ are similar

(c)

Suppose that $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ are similar

$\Rightarrow$ There exists an invertible matrix M such that $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}M^{-1}$

$$\Rightarrow M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}M^{-1} = MM^{-1} = I_{2\times 2} \neq \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$\Rightarrow \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ are not similar

#2

the eigenvalues of $A = -(a+b), 0$

$\lambda = -(a+b) \Rightarrow$ the eigenvector $X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 0 \Rightarrow$ the eigenvector $X = \begin{pmatrix} b \\ a \end{pmatrix}$

$AX = \lambda X$

$$\Rightarrow \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}\begin{pmatrix} 1 & b \\ -1 & a \end{pmatrix} = \begin{pmatrix} 1 & b \\ -1 & a \end{pmatrix}\begin{pmatrix} -(a+b) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix} = \begin{pmatrix} 1 & b \\ -1 & a \end{pmatrix}\begin{pmatrix} -(a+b) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ -1 & a \end{pmatrix}^{-1}$$

$$\Rightarrow e^{At} = \begin{pmatrix} 1 & b \\ -1 & a \end{pmatrix}\begin{pmatrix} e^{-(a+b)t} & 0 \\ 0 & e^{0t} \end{pmatrix}\begin{pmatrix} 1 & b \\ -1 & a \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a}{a+b}e^{-(a+b)t} + \frac{b}{a+b} & \frac{-b}{a+b}e^{-(a+b)t} + \frac{b}{a+b} \\ \frac{-a}{a+b}e^{-(a+b)t} + \frac{a}{a+b} & \frac{b}{a+b}e^{-(a+b)t} + \frac{a}{a+b} \end{pmatrix}$$

#3

(a)

$$A = \begin{pmatrix} -\frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & -\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -\pi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$\Rightarrow \cos A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \cos(-\pi) & 0 \\ 0 & \cos(0) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$\Rightarrow \cos A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(b)

A is diagonalizable

$$\text{let } A = S\Lambda S^{-1} = S\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}S^{-1}$$

$$\cos^2(A) = S\begin{pmatrix} \cos^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \cos^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \cos^2(\lambda_n) \end{pmatrix}S^{-1}$$

$$\sin^2(A) = S\begin{pmatrix} \sin^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \sin^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sin^2(\lambda_n) \end{pmatrix}S^{-1}$$

$$\Rightarrow \cos^2(A)+\sin^2(A)$$

$$= S\begin{pmatrix} \cos^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \cos^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \cos^2(\lambda_n) \end{pmatrix} S^{-1} + S\begin{pmatrix} \sin^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \sin^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sin^2(\lambda_n) \end{pmatrix} S^{-1}$$

$$= S(\begin{pmatrix} \cos^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \cos^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \cos^2(\lambda_n) \end{pmatrix} + \begin{pmatrix} \sin^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \sin^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sin^2(\lambda_n) \end{pmatrix})S^{-1}$$

$$= S\begin{pmatrix} \cos^2(\lambda_1)+\sin^2(\lambda_1) & 0 & \cdots & 0 \\ 0 & \cos^2(\lambda_2)+\sin^2(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \cos^2(\lambda_n)+\sin^2(\lambda_n) \end{pmatrix} S^{-1}$$

$$= SI_nS^{-1} = I_n$$

#4

(a)

*1

$<x,y>= x^TA^TAy = x^T\|A\|y = \|A\|x^Ty$

$<y,x>= y^TA^TAx = y^T\|A\|x = \|A\|y^Tx$

$\because x^Ty = y^Tx$

$\therefore <x,y>= \|A\|x^Ty = \|A\|y^Tx =<y,x>$

*2

$<x,y+t>= x^TA^TA(y+t) = x^TA^TAy + x^TA^TAt =<x,y>+<x,t>$

*3

$<cx,y>= (cx)^TA^TAy = c(x^TA^TAy) = c<x,y>$

*4

$<x,x>= x^TA^TAx = x^T\|A\|x = \|A\|x^Tx = \|A\|\|x\|$

$\|A\|,\|x\| > 0 \Rightarrow <x,x> \quad > 0$

by $1,2,3,4 \Rightarrow <x,y>= x^TA^TAy$ satisfies the four axioms of inner product spaces.

(b)

show that $A^*B=0 \Rightarrow C(A)\perp C(B)$

let $A=[A_1,A_2,\ldots,A_n]$; $B=[B_1,B_2,\ldots B_n]$

the size of $A_1,A_2,\ldots,A_n,B_1,B_2,\ldots B_n$ are $m\times 1$

$A^*B=0$

$$\Rightarrow \begin{pmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{pmatrix}[B_1 \quad B_2 \quad B_3 \quad B_4]=0$$

$$\Rightarrow \begin{pmatrix} A_1^*B_1 & A_1^*B_2 & \cdots & A_1^*B_n \\ A_2^*B_1 & A_2^*B_2 & \cdots & A_2^*B_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^*B_1 & A_1^*B_n & \cdots & A_n^*B_n \end{pmatrix}=0$$

$\Rightarrow A_1^*B_1=\cdots=A_1^*B_n=A_2^*B_1=A_2^*B_n=\cdots=A_n^*B_n=0$

$\Rightarrow C(A)\perp C(B)$

show that $C(A)\perp C(B) \Rightarrow A^*B=0$

let $A=[A_1,A_2,\ldots,A_n]$; $B=[B_1,B_2,\ldots B_n]$

the size of $A_1,A_2,\ldots,A_n,B_1,B_2,\ldots B_n$ are $m\times 1$

$\because C(A)\perp C(B)$

$\therefore <A_1,B_1>=<A_1,B_2>=\cdots=<A_1,B_n>=<A_2,B_1>=\cdots=<A_n,B_n>=0$

$\Rightarrow A_1^*B_1=\cdots=A_1^*B_n=A_2^*B_1=A_2^*B_n=\cdots=A_n^*B_n=0$

$$A^*B=\begin{pmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{pmatrix}[B_1 \quad B_2 \quad B_3 \quad B_4]=\begin{pmatrix} A_1^*B_1 & A_1^*B_2 & \cdots & A_1^*B_n \\ A_2^*B_1 & A_2^*B_2 & \cdots & A_2^*B_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^*B_1 & A_1^*B_n & \cdots & A_n^*B_n \end{pmatrix}=0$$

$\Rightarrow C(A)\perp C(B) \Leftrightarrow A^*B=0$

#5

$$A=\begin{pmatrix}1&1&2\\1&2&3\\1&3&4\end{pmatrix}\Rightarrow R_{23}^{(-1)}\Rightarrow\begin{pmatrix}1&1&2\\1&2&3\\0&1&1\end{pmatrix}\Rightarrow R_{12}^{(-1)}\Rightarrow\begin{pmatrix}1&1&2\\0&1&1\\0&1&1\end{pmatrix}$$

$$\Rightarrow R_{23}^{(-1)},R_{21}^{(-1)}\Rightarrow\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}$$

$$\Rightarrow C(A)=\text{span}\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix}\}\Rightarrow P_1=\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}\{\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}^T\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}\}^{-1}\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}^T=\begin{pmatrix}\frac{5}{6}&\frac{1}{3}&\frac{-1}{6}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{-1}{6}&\frac{1}{3}&\frac{5}{6}\end{pmatrix}$$

$$N(A)=\text{span}\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\}\Rightarrow P_2=\begin{pmatrix}-1\\-1\\1\end{pmatrix}\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}^T\begin{pmatrix}-1\\-1\\1\end{pmatrix}\}^{-1}\begin{pmatrix}-1\\-1\\1\end{pmatrix}^T=\begin{pmatrix}\frac{1}{3}&\frac{1}{3}&\frac{-1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{-1}{3}\\\frac{-1}{3}&\frac{-1}{3}&\frac{1}{3}\end{pmatrix}$$

$$C(A^T)=\text{span}\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\}\Rightarrow P_3=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}\{\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}^T\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}\}^{-1}\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}^T=\begin{pmatrix}\frac{2}{3}&\frac{-1}{3}&\frac{1}{3}\\\frac{-1}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\end{pmatrix}$$

$$N(A^T)=\text{span}\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\}\Rightarrow P_4=\begin{pmatrix}1\\-2\\1\end{pmatrix}\{\begin{pmatrix}1\\-2\\1\end{pmatrix}^T\begin{pmatrix}1\\-2\\1\end{pmatrix}\}^{-1}\begin{pmatrix}1\\-2\\1\end{pmatrix}^T=\begin{pmatrix}\frac{1}{6}&\frac{-1}{3}&\frac{1}{6}\\\frac{-1}{3}&\frac{2}{3}&\frac{-1}{3}\\\frac{1}{6}&\frac{-1}{3}&\frac{1}{6}\end{pmatrix}$$

#6

(a)

the dominant eigenvalue=5

the corresponding eigenvector of A= $\begin{pmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{pmatrix}$

(b)

When k=3 , we can get the dominant eigenvalue and corresponding eigenvector easily.
When k=2.1 , we can still get the dominant eigenvalue and corresponding eigenvector.
But it takes more time than k=3

The closer the largest absolute eigenvalue and the second largest absolute eigenvalue are, the more time it takes to get the dominant eigenvalue and corresponding eigenvector by power method.

When $k=2$,
we can't get the dominant eigenvalue and corresponding eigenvector by power method.

let λ_1 is the largest absolute eigenvalue
λ_2 is the second largest absolute eigenvalue
$x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ $\{v_1, v_2, \ldots, v_n\}$ are the eigenvectors of B

$$x_{k+1} = \frac{y}{\|y\|} = \frac{B\,x_k}{\|B\,x_k\|} = \frac{B^{k+1}x_0}{\|B^{k+1}x_0\|} = \frac{B^{k+1}(c_1v_1 + c_2v_2 + \cdots + c_nv_n)}{\|B^{k+1}(c_1v_1 + c_2v_2 + \cdots + c_nv_n)\|} = \frac{c_1B^{k+1}v_1 + c_2B^{k+1}v_2 + \cdots + c_nB^{k+1}v_n}{\|c_1B^{k+1}v_1 + c_2B^{k+1}v_2 + \cdots + c_nB^{k+1}v_n\|}$$

$$= \frac{c_1\lambda_1^{k+1}v_1 + c_2\lambda_2^{k+1}v_2 + \cdots + c_n\lambda_n^{k+1}v_n}{\|c_1\lambda_1^{k+1}v_1 + c_2\lambda_2^{k+1}v_2 + \cdots + c_n\lambda_n^{k+1}v_n\|} = \frac{c_1\lambda_1^{k+1}(v_1 + \frac{c_2}{c_1}\frac{\lambda_2^{k+1}}{\lambda_1^{k+1}}v_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^{k+1}}{\lambda_1^{k+1}}v_n)}{\|c_1\lambda_1^{k+1}(v_1 + \frac{c_2}{c_1}\frac{\lambda_2^{k+1}}{\lambda_1^{k+1}}v_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^{k+1}}{\lambda_1^{k+1}}v_n)\|}$$

$$= \pm\frac{(v_1 + \frac{c_2}{c_1}\frac{\lambda_2^{k+1}}{\lambda_1^{k+1}}v_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^{k+1}}{\lambda_1^{k+1}}v_n)}{\|v_1 + \frac{c_2}{c_1}\frac{\lambda_2^{k+1}}{\lambda_1^{k+1}}v_2 + \cdots + \frac{c_n}{c_1}\frac{\lambda_n^{k+1}}{\lambda_1^{k+1}}v_n\|}$$

if $\lambda_1 > \lambda_2, \ldots, \lambda_n \Rightarrow x_{k+1} = \pm\frac{v_1}{\|v_1\|} \Rightarrow$ We can get λ_1 easily.

if λ_1 and λ_2 are very close $\Rightarrow x_{k+1} \to \pm\frac{v_1}{\|v_1\|}$ when $k \to \infty$

$\Rightarrow$ It takes much time to get λ_1

if $\lambda_1 = \lambda_2 > \lambda_3, \ldots, \lambda_n \Rightarrow \{v_1, v_2, \ldots, v_n\}$ are dependent

$\Rightarrow$ B can't be diagonalizable

$\Rightarrow x_0$ can't be represented by $\{v_1, v_2, \ldots, v_n\}$

$\Rightarrow$ We can't get λ_1 by power method