Solutions to problem set 7

#1 (10pts)

(a)

$2+3+2+2+2+1=12$

the size of A is 12×12

(b)

$\text{rank}(A)=1+3+2+2+2+1=11$

(c)

$\lambda=0,0,2,2,2,2,2,3,3,3,3,3$

(d)

$\lambda=0$

algebraic multiplicities $=2$

geometric multiplicities $=12-\text{rank}(A)=1$

$\lambda=2$

algebraic multiplicities $=5$

geometric multiplicities $=12-\text{rank}(A-2I)=12-10=2$

$\lambda=3$

algebraic multiplicities $=5$

geometric multiplicities $=12-\text{rank}(A-5I)=12-9=3$

(e)

$$\text{let } e_1=\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}\ e_2=\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},\ldots,e_k=\begin{pmatrix}a_1\\\vdots\\a_j\\\vdots\\a_{12}\end{pmatrix}\ \text{if}\begin{cases}k=j\to a_j=1\\k\neq j\to a_j=0\end{cases}$$

$\Rightarrow$ the eigenvectors of J are $e_1,e_3,e_6,e_8,e_{10},e_{12}$ $\quad (e_k\in\mathbb{R}^{12},k=1,2,\ldots,12)$

(f)

$$\text{trace}(A)=\sum_{i=1}^{12}\lambda_i=0+0+2+2+2+2+2+3+3+3+3+3=25$$

#2 (20pts)

(a)

let $Ux=\lambda x$

$1.\ \|Ux\|^2=<Ux,Ux>=<\lambda x,\lambda x>=\lambda\bar{\lambda}<x,x>=|\lambda|^2<x,x>$

$2.\ <Ux,Ux>=(Ux)^*(Ux)=x^*U^*Ux=x^*U^{-1}Ux=x^*x=<x,x>$

by 1. and 2. $<Ux,Ux>=|\lambda|^2<x,x>=<x,x>$

$\Rightarrow|\lambda|^2=1\Rightarrow|\lambda|=1$

(b)

λ is the eigenvalue of U

$\Rightarrow Ux = \lambda x$

$\Rightarrow (Ux)^* = (\lambda x)^*$

$\Rightarrow x^* U^* = \overline{\lambda} x^*$

$\Rightarrow x^* U^{-1} = \overline{\lambda} x^*$

$\Rightarrow x^* = \overline{\lambda} x^* U = \overline{\lambda}(U^* x)^*$

$\Rightarrow x = \lambda U^* x$

$\Rightarrow U^* x = \frac{1}{\lambda} x$

$\Rightarrow \frac{1}{\lambda}$ is the eigenvalue of U^*

(c)

$\left|\det U^*\right| = \overline{\left|\det U\right|}$

$\Rightarrow \left|\det U^{-1}\right| = \overline{\left|\det U\right|}$

$\Rightarrow \frac{1}{\left|\det U\right|} = \overline{\left|\det U\right|}$

$\Rightarrow \left|\det U\right| \overline{\left|\det U\right|} = 1$

$\Rightarrow \left|\det U\right|^2 = 1$

$\Rightarrow \left|\det U\right| = 1$

(d)

$U^* U = I$

$$\Rightarrow \begin{pmatrix} -\overline{col_1}- \\ -\overline{col_2}- \\ \vdots \\ -\overline{col_n}- \end{pmatrix} \begin{pmatrix} | & | & & | \\ col_1 & col_2 & \cdots & col_n \\ | & | & & | \end{pmatrix} = I$$

$$\Rightarrow \overline{col_i^T} col_j = \begin{cases} 1 \ , \ \text{if} \ \ i = j \\ 0 \ , \ \text{if} \ \ i \neq j \end{cases}$$

$\Rightarrow$ the column vectors of U form an orthonormal basis for $\mathbb{C}^n$

(e)

$\because$ A is similar to U

$\therefore$ There exists an invertible matrix M such that $A = MUM^{-1}$ and $A^{-1} = MU^{-1}M^{-1}$

$A^* = \overline{A^T} = \overline{(M^{-1})^T U^T M^T} = \overline{(M^T)^{-1}}\,\overline{U^T}\,\overline{M^T}$

let $N = \overline{M^T} \Rightarrow A^* = N^{-1}\overline{U^T}N = N^{-1}U^*N = N^{-1}U^{-1}N = N^{-1}(M^{-1}A^{-1}M)N$

let $T = MN$, $T^{-1} = N^{-1}M^{-1}$

$\Rightarrow A^* = T^{-1}A^{-1}T$

$\Rightarrow A^*$ is similar to A^{-1}

#3 (20pts)

(a)

$(I-A)^*(I-A)=(I^*-A^*)(I-A)=I^*I-I^*A-A^*I+A^*A=I-A-A^*+A^*A$

$(I-A)(I-A)^*=(I-A)(I^*-A^*)=II^*-IA^*-AI^*+AA^*=I-A^*-A+AA^*$

$\because A^*A=AA^*$

$\therefore I-A-A^*+A^*A=I-A^*-A+AA^*$

$\Rightarrow (I-A)^*(I-A)=(I-A)(I-A)^*$

$\Rightarrow I-A$ is normal

(b)

$\|Ax\|^2=(Ax)^*(Ax)=x^*A^*Ax=x^*AA^*x=(A^*x)^*(A^*x)=\|A^*x\|^2$

$\Rightarrow \|Ax\|=\|A^*x\|$

(c)

$(A-\lambda I)^*(A-\lambda I)=(A^*-\bar{\lambda}I)(A-\lambda I)=A^*A-\lambda A^*-\bar{\lambda}A+\lambda^2 I$

$(A-\lambda I)(A-\lambda I)^*=(A-\lambda I)(A^*-\bar{\lambda}I)=AA^*-\lambda A^*-\bar{\lambda}A+\lambda^2 I$

$\because A^*A=AA^*$

$\therefore (A-\lambda I)^*(A-\lambda I)=(A-\lambda I)(A-\lambda I)^*$

$\Rightarrow (A-\lambda I)$ is normal

from the result in (b)

$\|(A-\lambda I)x\|=\|(A-\lambda I)^*x\|$

$(A-\lambda I)^*=A^*-\bar{\lambda}I^*=A^*-\bar{\lambda}I$

$\Rightarrow \|(A-\lambda I)x\|=\|(A^*-\bar{\lambda}I)x\|$

(d)

from the result in (c)

$\|(A-\lambda I)x\|=\|(A^*-\bar{\lambda}I)x\|$

$\Rightarrow \|(A-\lambda I)x\|=0=\|(A^*-\bar{\lambda}I)x\|$

$\Rightarrow (A^*-\bar{\lambda}I)x=A^*x-\bar{\lambda}Ix=[0]$

$\Rightarrow A^*x=\bar{\lambda}Ix$

(e)

let $Ax_1=\lambda_1 x_1 \quad Ax_2=\lambda_2 x_2$

$\lambda_1<x_1,x_2>=<\lambda_1 x_1,x_2>=<Ax_1,x_2>=(Ax_1)^*x_2=x_1^*A^*x_2=x_1{}^*\bar{\lambda}_2 x_2=<x_1,\bar{\lambda}_2 x_2>=\lambda_2<x_1,x_2>$

$\Rightarrow \lambda_1<x_1,x_2>-\lambda_2<x_1,x_2>=(\lambda_1-\lambda_2)<x_1,x_2>=0$

$\Rightarrow \lambda_1,\lambda_2$ are distinct eigenvalue $\Rightarrow \lambda_1\neq\lambda_2$

$\Rightarrow <x_1,x_2>=0$

$\Rightarrow$ the eigenvectors belonging to distinct eigenvalues are orthogonal

#4 (10pts)

show that A is normal if $HS=SH$

$AA^* = (H+S)(H+S)^*$

$\Rightarrow AA^* = (H+S)(H^*+S^*)$

$\Rightarrow AA^* = HH^* + SH^* + HS^* + SS^*$

$\because SH = HS \ , \ H^* = H \ , \ S^* = -S$

$\therefore SH^* = SH = HS = H^*S \ , \ HS^* = -HS = -SH = S^*H$

$\Rightarrow AA^* = HH^* + SH^* + HS^* + SS^* = H^*H + H^*S + S^*H + S^*S$

$\Rightarrow AA^* = H^*(H+S) + S^*(H+S)$

$\Rightarrow AA^* = (H^*+S^*)(H+S) = (H+S)^*(H+S) = A^*A$

show that $HS=SH$ if A is normal

$AA^* = (H+S)(H+S)^*$

$A^*A = (H+S)^*(H+S)$

$\because$ A is normal

$\therefore AA^* = A^*A$

$\Rightarrow (H+S)(H+S)^* = (H+S)^*(H+S)$

$\Rightarrow (H+S)(H^*+S^*) = (H^*+S^*)(H+S)$

$\Rightarrow HH^* + SH^* + HS^* + SS^* = H^*H + H^*S + S^*H + S^*S$

$\Rightarrow H^* = H \ , \ S^* = -S \Rightarrow HH + SH - HS - SS = HH + HS - SH - SS$

$\Rightarrow SH - HS = HS - SH$

$\Rightarrow SH = HS$

#5 (15pts)

(a)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \text{rref} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R(A) = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}$$

when $x \in$ Row space of A , $\|x\|$ has minimum $\|x_0\|$ such that $Ax_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(You can refer to http://ccjou.twbbs.org/blog/wp-content/uploads/2009/05/powsol-may-4-09.pdf)

$$\text{let}\quad x_0 = c_1\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$Ax_0 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&2&3&4\\0&1&1&1\\0&1&1&1\end{pmatrix}\{c_1\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&2&3&4\\0&1&1&1\\0&1&1&1\end{pmatrix}\{\begin{pmatrix}1&0\\0&1\\1&1\\2&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}\} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}12&9\\3&3\\3&3\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-\frac{2}{3}\\1\end{pmatrix}$$

$$\Rightarrow x_0 = (-\frac{2}{3})\times\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + \begin{pmatrix}0\\1\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{-2}{3}\\1\\\frac{1}{3}\\\frac{-1}{3}\end{pmatrix}$$

(b)

$$C(A) = \text{span}\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\}$$

$$\because b = \begin{pmatrix}0\\1\\0\end{pmatrix} \notin c(A)$$

$$\therefore \text{project } b \text{ to } C(A) \text{ to minimize } \left\| Ax - \begin{pmatrix}0\\1\\0\end{pmatrix} \right\|$$

$$p = \begin{pmatrix}1 & 2\\0 & 1\\0 & 1\end{pmatrix}(\begin{pmatrix}1 & 0 & 0\\2 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\\0 & 1\end{pmatrix})^{-1}\begin{pmatrix}1 & 0 & 0\\2 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 0.5 & 0.5\\0 & 0.5 & 0.5\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0.5\\0.5\end{pmatrix}$$

$$\text{find minimum } x_0 \text{ such that } Ax_0 = \begin{pmatrix}0\\0.5\\0.5\end{pmatrix}$$

when $x \in$ Row space of A , $\|x\|$ has minimum $\|x_0\|$

$$\text{let } x_0 = c_1\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$Ax_0 = \begin{pmatrix}0\\0.5\\0.5\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 2 & 3 & 4\\0 & 1 & 1 & 1\\0 & 1 & 1 & 1\end{pmatrix}\{c_1\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\} = \begin{pmatrix}0\\0.5\\0.5\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 2 & 3 & 4\\0 & 1 & 1 & 1\\0 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\1 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0.5\\0.5\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}12 & 9\\3 & 3\\3 & 3\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0.5\\0.5\end{pmatrix}$$

$$\Rightarrow c_1 = -0.5 \text{ , } c_2 = \frac{2}{3}$$

$$x_0 = -0.5\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-0.5\\ \frac{2}{3}\\ \frac{1}{6}\\ \frac{-1}{3}\end{pmatrix}$$

#6 (25pts)

(a)

let $u_1 = 1$

$$u_2 = x - \frac{<x,1>}{<1,1>}1 = x - \frac{1}{2}$$

$$\|u_2\|^2 = <u_2,u_2> = \int_0^1 (x-\frac{1}{2})^2 dx = \frac{1}{12} \Rightarrow \|u_2\| = \frac{1}{\sqrt{12}}$$

$$\Rightarrow \text{let } w_2 = \sqrt{12}(x-\frac{1}{2}) \;,\; w_1 = 1$$

$\{w_1, w_2\}$ is an orthonormal basis for W

(b)

$$y = \frac{<x^3,w_1>}{<w_1,w_1>}w_1 + \frac{<x^3,w_2>}{<w_2,w_2>}w_2$$

$$<x^3,w_1> = \int_0^1 x^3 dx = \frac{1}{4}$$

$$<x^3,w_2> = \int_0^1 x^3[\sqrt{12}(x-\frac{1}{2})]dx = \frac{3}{20}\sqrt{3}$$

$$\Rightarrow y = \frac{1}{4}w_1 + \frac{3}{20}\sqrt{3}w_2 = \frac{1}{4} + \frac{3}{20}\sqrt{3}[\sqrt{12}(x-\frac{1}{2})] = 0.9x - 0.2$$

(c)

let w_3 is the basis for $W^{\perp}$

$$w_3 = x^2 - \frac{<x^2,w_1>}{<w_1,w_1>}w_1 - \frac{<x^2,w_2>}{<w_2,w_2>}w_2$$

$$<x^2,w_1> = \int_0^1 x^2 dx = \frac{1}{3}$$

$$<x^2,w_2> = \int_0^1 x^2[\sqrt{12}(x-\frac{1}{2})]dx = \frac{1}{6}\sqrt{3}$$

$$\Rightarrow w_3 = x^2 - \frac{1}{3}w_1 - \frac{1}{6}\sqrt{3}w_2$$

$$\Rightarrow w_3 = x^2 - x + \frac{1}{6}$$