Solutions to problem set 8

#1

(a)

No!

*counter example

let $A=\begin{pmatrix}2 & 0\\ 0 & \frac{1}{2}\end{pmatrix}$

$A^{-1}=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 2\end{pmatrix}$

the eigenvalue of A are 0.5 , 2

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}2 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 2\end{pmatrix}$$

$\Rightarrow S^{-1}AS=A^{-1}$

$\Rightarrow A$ and A^{-1} are similar but the eigenvalue of A are not $+1,-1$

(b)

the eigenvalue of $A=1,1,1$

the eigenvalue of $B=1,1,1$

the eigenvalue of $C=1,-1,3$

$\Rightarrow A$ and C don't have the same eigenvalue $\Rightarrow A$ and C are not similar

$\Rightarrow B$ and C don't have the same eigenvalue $\Rightarrow B$ and C are not similar

$\lambda_A=1,1,1$

$$\lambda_{A1}=1\Rightarrow\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\lambda_{A2}=1\Rightarrow\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\lambda_{A2}=1\Rightarrow\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Rightarrow M_A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\Rightarrow M_A^{-1}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$\Rightarrow J_A=M_A^{-1}AM_A=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$$

$\lambda_B = 1,1,1$

$$\lambda_{B1} = 1 \Rightarrow \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_{B2} = 1 \Rightarrow \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\lambda_{B3} = 1 \Rightarrow \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow M_B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow M_B^{-1} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow J_B = M_B^{-1} B M_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\because J_A = J_B$

$\therefore$ only A and B are similar

#2

the eigenvalue of $A = 2,2,1$

let $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)X_1 = 0$

$$\Rightarrow \begin{pmatrix} 2 & -7 & -2 \\ 1 & -3 & -1 \\ -3 & 9 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

let $\lambda_2 = 2 \Rightarrow (A - \lambda_2 I)X_2 = 0$

$$\Rightarrow \begin{pmatrix} 1 & -7 & -2 \\ 1 & -4 & -1 \\ -3 & 9 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

let $\lambda_3 = 2 \Rightarrow (A - \lambda_3 I)X_3 = X_2$

$$\Rightarrow \begin{pmatrix} 1 & -7 & -2 \\ 1 & -4 & -1 \\ -3 & 9 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -3 & 0 \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & -1 \end{pmatrix}$$

$$J = M^{-1} A M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

#3

(a)

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}$$

its characteristic polynomial is $\det(A-\lambda I)=\det(\begin{pmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda & 1 \\ 1 & 0 & \cdots & 0 & -\lambda \end{pmatrix})$

$$=(-\lambda)\det(\begin{pmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda & 1 \\ 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix}_{(n-1)\times(n-1)})+(-1)\times\det(\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda & 1 \\ 1 & 0 & 0 & \cdots & -\lambda \end{pmatrix}_{(n-1)\times(n-1)})$$

by exchanging the rows $(n-2)$ times we can get

$$\det(\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda & 1 \\ 1 & 0 & \cdots & 0 & -\lambda \end{pmatrix}_{(n-1)\times(n-1)})=(-1)^{n-2}\det(\begin{pmatrix} 1 & 0 & \cdots & 0 & -\lambda \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & -\lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda & 1 \end{pmatrix}_{(n-1)\times(n-1)})=(-1)^{n-2}$$

$\Rightarrow \det(A-\lambda I)=(-\lambda)^n+(-1)^{n-1}$

let $\det(A-\lambda I)=0 \Rightarrow (-\lambda)^n+(-1)^{n-1}=0$

$\Rightarrow (-1)^n[\lambda^n-1]=0 \Rightarrow \lambda^n=1$

The solution of $\det(A-\lambda I)=0$ lies on the unit circle of the complex plane, and there are n distinct eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$

$$\Rightarrow \text{The Jordan form of } A=\begin{pmatrix} J(\lambda_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J(\lambda_n) \end{pmatrix}=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

(b)

Assume $J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$

$\because$ eigenvalue of J^{-1} is $\dfrac{1}{\lambda}$ when $\lambda \neq 0$

$\therefore$ Jordan form of $J^{-1} = J(\dfrac{1}{\lambda}) = \begin{pmatrix} 1/\lambda & 1 & 0 & \cdots & 0 \\ 0 & 1/\lambda & 1 & \ddots & \vdots \\ 0 & 0 & 1/\lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & 1/\lambda \end{pmatrix}$

$\because$ eigenvalue of J^2 is λ^2

$\therefore$ Jordan form of $J^2 = J(\lambda^2) = \begin{pmatrix} \lambda^2 & 1 & 0 & \cdots & 0 \\ 0 & \lambda^2 & 1 & \ddots & \vdots \\ 0 & 0 & \lambda^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda^2 \end{pmatrix}$

(c)

For any nonsingular matrix A, we can decompose it using its Jordan Form:

$$A = M\, J_A\, M^{-1}$$

(1) For the case that A has distinct eigenvalues λ_1 & λ_2:

A is diagonalizable, thus it is trivial to find out a matrix B such that B^2=A.

(2) For the case that A is not diagonalizable:

Assume $\quad J_A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$\because$ We are trying to find B such that B^2=A

$\therefore$ It is reasonable to guess that the Jordan Form of B is

$$J_B = \begin{bmatrix} \lambda_B & 1 \\ 0 & \lambda_B \end{bmatrix}, \text{where } \lambda_B^2 = \lambda\,.$$

Therefore,

$$J_B^2 = \begin{bmatrix} \lambda_B^2 & 2\lambda_B \\ 0 & \lambda_B^2 \end{bmatrix}$$

If we can find out a matrix N such that $J_B^2 = N\,J_A\,N^{-1}$, then there **must** exist a matrix $B = S\,J_B\,S^{-1}$, which satisfies

$$B^2 = S\,J_B^2\,S^{-1} = S(NJ_AN^{-1})S^{-1} = (SN)J_A(SN)^{-1} = M\,J_A\,M^{-1} = A.$$

<Why can we assure the existence of B by finding out N?>

Once N is found, we can find out a matrix S=MN^{-1}. That is, there must exist a matrix $B = S\,J_B\,S^{-1} = (MN^{-1})J_A(MN^{-1})^{-1}$ that satisfies B^2=A.

The problem is now reduced: Does N exist? How to find out N?

Since we know that $J_B^2 = N\,J_A\,N^{-1}$, we can find out N in an old-fashioned way: let $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and use algebraic method to solve a, b, c and d.

$$J_B^2 = N\,J_A\,N^{-1} \Rightarrow J_B^2 N = N\,J_A \Rightarrow \begin{bmatrix} \lambda_B^2 & 2\lambda_B \\ 0 & \lambda_B^2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_B^2 & 1 \\ 0 & \lambda_B^2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a\lambda_B^2 + 2c\lambda_B = a\lambda_B^2 \\ b\lambda_B^2 + 2d\lambda_B = a + b\lambda_B^2 \\ c\lambda_B^2 = c\lambda_B^2 \\ d\lambda_B^2 = d\lambda_B^2 \end{cases} \Rightarrow c = 0, d = \frac{a}{2\lambda_B} \Rightarrow N = \begin{bmatrix} a & b \\ 0 & \frac{a}{2\lambda_B} \end{bmatrix}$$

For simplicity, choose a=b=1 and we have $N = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2\lambda_B} \end{bmatrix}$.

Note that nonsingular matrix A guarantees that $\lambda_B \neq 0$, so there exists N such that $J_B^2 = N\,J_A\,N^{-1}$. By the comment mentioned above, we can conclude that there exists a matrix B such that B^2=A.

#4

$$J_m(\lambda)=\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}=\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 0 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}+\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}=\lambda I+J_m(0)$$

$$[J_m(\lambda)]^k=[\lambda I+J_m(0)]^k=\sum_{i=0}^{k}C_i^k\lambda^i I^i[J_m(0)]^{k-i}=\sum_{i=0}^{k}C_i^k\lambda^i[J_m(0)]^{k-i}$$

$$\text{when } i=k\Rightarrow C_i^k\lambda^i[J_m(0)]^{k-i}=\lambda^k I=\begin{pmatrix} \lambda^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda^k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda^k & 0 \\ 0 & \cdots & 0 & 0 & \lambda^k \end{pmatrix}$$

$$\text{when } i=k-1\Rightarrow C_i^k\lambda^i[J_m(0)]^{k-i}=C_{k-1}^k\lambda^{k-1}[J_m(0)]^1=C_1^k\lambda^{k-1}[J_m(0)]^1=\begin{pmatrix} 0 & C_1^k\lambda^{k-1} & 0 & \cdots & 0 \\ 0 & 0 & C_1^k\lambda^{k-1} & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & C_1^k\lambda^{k-1} \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$\text{when } i=k-2\Rightarrow C_i^k\lambda^i[J_m(0)]^{k-i}=C_{k-2}^k\lambda^{k-2}[J_m(0)]^2=C_2^k\lambda^{k-2}[J_m(0)]^2=\begin{pmatrix} 0 & 0 & C_2^k\lambda^{k-2} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & C_2^k\lambda^{k-2} \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$\vdots$$

$$\text{if } k>m-1\Rightarrow \text{when } i=m-1\Rightarrow C_i^k\lambda^i[J_m(0)]^{k-i}=C_{m-1}^k\lambda^{k-m+1}[J_m(0)]^{m-1}$$

$$=\begin{pmatrix} 0 & 0 & 0 & \cdots & C_{m-1}^k\lambda^{k-m+1} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow [J_m(\lambda)]^k=\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}^k=\begin{pmatrix} \lambda^k & C_1^k\lambda^{k-1} & C_2^k\lambda^{k-2} & \cdots & C_{m-1}^k\lambda^{k-m+1} \\ 0 & \lambda^k & C_1^k\lambda^{k-1} & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & C_2^k\lambda^{k-2} \\ \vdots & \ddots & \ddots & \lambda^k & C_1^k\lambda^{k-1} \\ 0 & \cdots & 0 & 0 & \lambda^k \end{pmatrix}$$

#5

(a)

consider

$x(t) = x_0 e^{At} e^{Bt}$

$y(t) = y_0 e^{(A+B)t}$

$x(0) = x_0 = y(0) = y_0$

let $z(t) = x_0 e^{Bt}$

$x'(t) = (x_0 e^{At} e^{Bt})' = [e^{At} z(t)]'$

$\Rightarrow x'(t) = Ae^{At} z(t) + Be^{At} z(t)$

$\Rightarrow x'(t) = Ax(t) + Bx(t)$

$\Rightarrow x'(t) = (A+B)x(t)$

$y'(t) = y_0 (A+B) e^{(A+B)t} = (A+B)y(t)$

$\because x(0) = y(0)$

$\therefore x(t) = y(t)$,for $t \geq 0$

$\Rightarrow x_0 e^{A} e^{B} = x(1) = y(1) = y_0 e^{(A+B)}$

$\Rightarrow e^{A} e^{B} = e^{(A+B)}$

(b)

$$\text{let } A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\text{let } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$$

$$BA = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow AB \neq BA$

$$e^{A} e^{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2} & 0 \\ 0 & e^{1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e^{1} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{1} & -e^{1} + e^{3} \\ e^{1} - 1 & -e^{1} + 1 + e^{3} \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$e^{A+B} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{0} & 0 \\ 0 & e^{3} \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{-1}{3}\begin{pmatrix} -2 - e^{3} & 2 - 2e^{3} \\ 1 - e^{3} & -1 - 2e^{3} \end{pmatrix}$$

$\Rightarrow e^{A} e^{B} \neq e^{A+B}$