Solution to problem set 9

#1

(a)

In general, we define

$q(A) = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I$

$q(\lambda) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0 I$

$q(A)X_A = c_n A^n X_A + c_{n-1} A^{n-1} X_A + \cdots + c_1 A X_A + c_0 X_A$

$q(A)X_A = c_n \lambda^n X_A + c_{n-1} \lambda^{n-1} X_A + \cdots + c_1 \lambda X_A + c_0 X_A = q(\lambda) X_A$

$p_A(\lambda) = 0$ doesn't mean that all the eigenvalue of A satisfy $p_A(\lambda) = 0$.

It means that all the eigenvalues of A satisfy $p(A) = 0$

(b)

First, in Cayley-Hamilton theorem, p(A) is an $n \times n$ matrix.

However, the right hand side of the above equation is the value of a determinant, which is a scalar

So they cannot be equated unless n = 1 (i.e. A is just a scalar).

Second, in the expression det(A-λI),

the variable λ actually occurs at the diagonal entries of the matrix (A-λI)

To illustrate, consider the characteristic polynomial in the previous example again:

let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\Rightarrow \det(A-\lambda I) = \det\left(\begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix}\right)$

If one substitutes the entire matrix A for λ in those positions, one obtains

$$\det\left(\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 1 & -2 \\ -3 & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 4 \end{pmatrix}\right)$$

in which the "matrix" expression is simply not a valid one

#2

$$\text{let } A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 1 & 1 & 3 & 4 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 4 \\ 1 & 1 & 3 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 1 & 3 & 4-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 1 & 3 & 4-\lambda \end{vmatrix}$$

$\Rightarrow \det(A - \lambda I) = (1-\lambda)(2-\lambda)[(3-\lambda)(4-\lambda) - 12] - 2[(3-\lambda)(4-\lambda) - 12] = \lambda^2(\lambda - 7)(\lambda - 3) = 0$

$\Rightarrow \lambda = 0, 0, 3, 7$

when $\lambda = 0$

$\text{Rank}(A - \lambda I) = \text{Rank}(A) = 3 \Rightarrow \text{Geometric Multiplicity} = 3$

$\text{Algebraic Multiplicity} = 2$

$\Rightarrow \text{Geometric Multiplicity} \neq \text{Algebraic Multiplicity}$

$\Rightarrow$ A can't be diagonalizable

characteristic polynomial: $p_A(\lambda) = \lambda^2(\lambda - 7)(\lambda - 3)$

if $q_A(\lambda) = \lambda(\lambda - 7)(\lambda - 3) \Rightarrow q_A(A) = A(A - 7I)(A - 3I) \neq 0$

if $q_A(\lambda) = \lambda^2(\lambda - 7)(\lambda - 3) \Rightarrow q_A(A) = A^2(A - 7I)(A - 3I) = 0$

$\Rightarrow$ minimum polynomial: $m_A(\lambda) = \lambda^2(\lambda - 7)(\lambda - 3)$

$$\Rightarrow \text{Its Jordan form}: J_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

#3

(a)

let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow A^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A^* = A$

$\Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = A^*$

$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an unitary and Hermitian matrix

(b)

let $B = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

$\Rightarrow B^* = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow B^* = -B$

$\Rightarrow B^{-1} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow B^{-1} = B^*$

$\Rightarrow B = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is an unitary and skew$-$Hermitian matrix

#4

(a)True pf: $(A + B)^* = A^* + B^* = A + B$

(b)False

counter example: let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $c = i \Rightarrow cA = iA = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

$\Rightarrow (cA)^* = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = -(cA) \neq cA$

$\Rightarrow$ cA isn't a Hermitian for every complex number c

(c)False pf: $(AB)^* = B^*A^* = BA \neq AB$

(d)True pf: $(ABA)^* = A^*B^*A^* = ABA$

(e)False counter example: let $A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Rightarrow A\neq I, A^2=I$

(f)True pf: $0=A^2=AA=A^*A=\|A\|^2\Rightarrow\|A\|=0\Rightarrow A=0$

(g)True pf: $(iA)^*=\bar{i}A^*=\bar{i}A=-iA$

(h)True

#5

(a)

$(AB)^*=B^*A^*=BA=AB$

$\Rightarrow AB$ is Hermitian matrix

(b) No!

counter example:

let $A=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\Rightarrow A$ is a Hermitian matrix

let $B=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}^{-1}\Rightarrow A\sim B$

$\Rightarrow B=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -2 & 4\end{pmatrix}=\begin{pmatrix}0 & 2\\ -1 & 3\end{pmatrix}$

$\Rightarrow B^*=\begin{pmatrix}0 & 2\\ -1 & 3\end{pmatrix}^*=\begin{pmatrix}0 & -1\\ 2 & 3\end{pmatrix}\neq B$

$\Rightarrow B$ isn't a Hermitian

#6

(a)

if we consider $a\in\mathbb{R}$ then $\bar{a}\in\mathbb{R}\Rightarrow\begin{pmatrix}1 & 1 & a\\ 1 & 1 & 1\\ \bar{a} & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 1 & a\\ 1 & 1 & 1\\ a & 1 & 1\end{pmatrix}$ is a real symmetric matrix

for any real symmetric matrix, if $\forall\lambda_i\geq 0$ then x^TAx is positive semidefinite

$$\begin{vmatrix}1-\lambda & 1 & a\\ 1 & 1-\lambda & 1\\ a & 1 & 1-\lambda\end{vmatrix}=(1-\lambda)^3+a+a-a^2(1-\lambda)-(1-\lambda)-(1-\lambda)=-\lambda^3+3\lambda^2+(a^2-1)\lambda-1-a^2+2a=0$$

$\lambda_1\lambda_2\lambda_3=-1-a^2+2a\geq 0\Rightarrow -(a-1)^2\geq 0\Rightarrow a=1$

$\Rightarrow$ when $a=1$, $\begin{pmatrix}1 & 1 & a\\ 1 & 1 & 1\\ \bar{a} & 1 & 1\end{pmatrix}$ is positive semidefinite if we consider $b \in \mathbb{R}$ then $\bar{b} \in \mathbb{R} \Rightarrow \begin{pmatrix} 1 & b & 0 \\ \bar{b} & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & 0 \\ b & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is a real symmetric matrix

for any real symmetric matrix , if $\forall \lambda_i \geq 0$ then $x^T A x$ is positive semidefinite

$$\begin{vmatrix} 1-\lambda & b & 0 \\ b & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - (1-\lambda) - b^2(1-\lambda) = -\lambda^3 + 3\lambda^2 + (b^2-2)\lambda - b^2 = 0$$

$\lambda_1 \lambda_2 \lambda_3 = -b^2 \geq 0 \Rightarrow b^2 \leq 0 \Rightarrow b = 0$

$\Rightarrow$ when $b = 0$, $\begin{pmatrix} 1 & b & 0 \\ \bar{b} & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is positive semidefinite

(b)

$x^* A^2 x = x^* A A x = x^* A^* A x = x^* \|A\|^2 x = \|A\|^2 x^* x = \|A\|^2 \|x\|^2 \geq 0$

$\Rightarrow \forall x$ such that $x^T A^2 x \geq 0 \Rightarrow A^2$ is positive semidefinite

(c)

$x^*(-A^2)x = x^*(-A)Ax = x^*(A^*)Ax = x^* \|A\|^2 x = \|A\|^2 x^* x = \|A\|^2 \|x\|^2 \geq 0$

$\Rightarrow \forall x$ such that $x^T(-A^2)x \geq 0 \Rightarrow -A^2$ is positive semidefinite

(d) No!

$(x^* A x)^* = x^* A^* x = x^*(-A)x = -x^* A x \Rightarrow x^* A x$ is a pure imaginary number for any $x \neq 0$

$x^* A x$ might not be 0

(e)

<1>No! counter example: let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow Ax = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq 0$

$\Rightarrow x^* A x = [1 \quad 0 \quad 0]\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

<2>Consider A is positive semidefinite for some $x \neq 0$

If we can proof that "$Ax \neq 0$ and $A \neq 0 \Rightarrow x^* A x \neq 0$"

then we can get the following conclusion.

If $x^*Ax=0$ and A is positive semidefinite for some $x \neq 0$ then it follow that $Ax=0$ or $A=0$

Pf : (Proof by Contradition)

for $Ax \neq 0$ and $A \neq 0$

let $x^* A x = 0$

$\Rightarrow x$ is orthogonal to Ax

$\Rightarrow x$ is orthogonal to $C(A)$

$\Rightarrow x \in$ left nullspace of A

$\Rightarrow x^* A = 0$

$\because$ A is positive semidefinite

$\therefore x^* A = x^* A^*$

$\Rightarrow x^* A^* = 0$

$\Rightarrow (Ax)^* = 0$

$\Rightarrow Ax = 0$

Contradiction!

$\Rightarrow$ If $Ax \neq 0$ and $A \neq 0$ then $x^* Ax \neq 0$

$\Rightarrow$ If $x^*Ax=0$ and A is positive semidefinite for some $x \neq 0$ then it follow that Ax=0 or A=0