Problem for the week of January 23, 2012

Let A and B be 5×7 and 7×6 matrices, respectively. If $\text{rank}A = 3$ and $\text{rank}B = 5$, find all possible values of $\text{rank}(AB)$.

Solution

我們從矩陣乘積 AB 的行空間分析著手。因爲 $C(AB) \subseteq C(A)$, 可知 $\text{dim}C(AB) \le \text{dim}C(A)$, 亦即 $\text{rank}(AB) \le \text{rank}A$。轉置不改變矩陣秩, 就有

$$\text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^TA^T) \le \text{rank}B^T = \text{rank}B$$

故得到 $\text{rank}(AB)$ 上界:

$$\text{rank}(AB) \le \min\{\text{rank}A, \text{rank}B\}$$

矩陣乘積可視爲變換 A 具有定義域 $C(B)$, 值域 $C(AB)$, 零空間 $C(B)\cap N(A)$。由秩–零度定理, 可得

$$\text{dim}C(B) = \text{dim}C(AB) + \text{dim}(C(B)\cap N(A))$$

改寫爲

$$\text{rank}(AB) = \text{rank}B - \text{dim}(C(B)\cap N(A))$$

但是 $C(B)\cap N(A) \subseteq N(A)$, 也就有 $\text{dim}(C(B)\cap N(A)) \le \text{dim}N(A) = n - \text{rank}A$, 故得到 $\text{rank}(AB)$ 下界:

$$\text{rank}(AB) \ge \text{rank}B + \text{rank}A - n$$

最後代入數值 $n = 7$, $\text{rank}A = 3$, $\text{rank}B = 5$, 即得 $1 \le \text{rank}(AB) \le 3$。 □