**Linear Algebra**

**Problem Set 1** **Spring 2015**

Due Tuesday, 10 March 2015 at 12:00 PM in EE106. This problem set covers Lecture 1-4. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (10pts) Let $A=\begin{bmatrix}1&2&2\\2&5&4\\3&7&6\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$. Describe the set of all **b** for which $A\mathbf{x}=\mathbf{b}$ does have a solution.
2. (10pts) Write down the 3 by 3 elementary matrices that produce these elimination steps:
   (a) $E_{21}$: subtract 5 times row1 from row2.
   (b) $E_{23}$: subtract $-7$ times row3 from row2.
   (c) $P$: exchange row1 and row2, then row1 and row3 (two steps).
   (d) $S$: multiply row3 by 4, then row1 by 6 (two steps).
3. (10pts) Let
   $$A=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}.$$
   Find elimination matrices $E_{21}$ then $E_{32}$ then $E_{43}$ to change $A$ into an echelon form $U$, i.e., $E_{43}E_{32}E_{21}A=U$.
4. (10pts) Consider the following three systems where the coefficients are the same for each system, but the right-hand sides are different:
   $$\begin{aligned}x+2y+z&=1\,|\,0\,|\,0\\ y+4z&=0\,|\,1\,|\,0\\ x+2y&=0\,|\,0\,|\,1\end{aligned}$$
   Solve all three systems at one time by performing Gaussian elimination on an augmented matrix of the form $\left[A|\mathbf{b}_1|\mathbf{b}_2|\mathbf{b}_3\right]$. What is the 3 by 3 matrix $X$ such that $XA=I$?
5. (15pts) The three types of elementary row operations are not independent: the row exchange operation can be accomplished by a sequence of the other two types of row operations, namely, replacing (i.e., subtracting $k\times$ row*i* from row*j*) and scaling (i.e., multiplying row*i* by *k*, $k\neq 0$). Apply a sequence of replacing and

scaling operations to the 3 by 3 identify matrix $I$ to obtain the permutation matrix:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

6. (10pts) True or false. Give a specific example when false:
   (a) If columns 1 and 2 of $A$ are the same, so are columns 1 and 2 of $AB$.
   (b) If rows 1 and 3 of $A$ are the same, so are rows 1 and 3 of $AB$.
   (c) If rows 2 and 4 of $A$ are the same, so are rows 2 and 4 of $ABC$.
7. (15pts) Without using Gaussian elimination, find $X$ by inspection. Identify the size of $X$ first.
   (a) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$
   (b) $$X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$
8. (20pts) Section 2.4, Problem 36

   To prove that $(AB)C = A(BC)$, use the column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of $B$. First suppose that $C$ has only one column $\mathbf{c}$ with entries $c_1, \ldots, c_n$:

   $AB$ has columns $A\mathbf{b}_1, \ldots, A\mathbf{b}_n$ and then $(AB)\mathbf{c}$ equals $c_1 A\mathbf{b}_1 + \ldots + c_n A\mathbf{b}_n$.

   $B\mathbf{c}$ has one column $c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ and then $A(B\mathbf{c})$ equals $A(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n)$.

   *Linearity* gives equality of those two sums. This proves $(AB)\mathbf{c} = A(B\mathbf{c})$. The same is true for all other _____of $C$. Therefore $(AB)C = A(BC)$. Apply to inverses:

   If $BA=I$ and $AC=I$, prove that the left-inverse $B$ equals the right-inverse $C$. Show every step of your proof.