Linear Algebra

Problem Set 10 **Spring 2015**

Due Thursday, 11 June 2015 at 4:20 PM in EE106. This problem set covers Lecture 39-42. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (10pts) Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

2. (10pts) Even if A is rectangular, the block matrix $B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ is symmetric.

   Suppose $B\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}$, $A^T\mathbf{y} = \lambda\mathbf{z}$ and $A\mathbf{z} = \lambda\mathbf{y}$.

   (a) Show that $-\lambda$ is also an eigenvalue. Find the corresponding eigenvector.

   (b) Show that $A^T A\mathbf{z} = \lambda^2\mathbf{z}$.

3. (30pts) True or false, with a reason or counterexample if false.

   (a) A matrix with real eigenvalues and eigenvectors is symmetric.

   (b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

   (c) The inverse of a symmetric matrix is symmetric.

   (d) If A is symmetric and similar to B, then B is symmetric.

   (e) The eigenvector matrix S of a symmetric matrix is symmetric.

   (f) Every positive definite symmetric matrix is nonsingular.

   (g) The inverse of a positive definite matrix is positive definite.

   (h) The sum of two positive definite matrices is positive definite.

   (i) A diagonal matrix with positive diagonal entries is positive definite.

   (j) The only positive definite projection matrix is $P = I$.

4. (10pts) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, positive definite?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

5. (15pts) Suppose A is real antisymmetric, i.e., $A^T = -A$. Prove the following statements.

   (a) $\mathbf{x}^T A\mathbf{x} = 0$ for every real vector $\mathbf{x}$.

(b) The eigenvalues of A are pure imaginary (including zero).

(c) The determinant of A is positive or zero.

6. (10pts) For what numbers c and d will A and B have positive eigenvalues? Test the 3 determinants:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 & 0 \\ 3 & d & 4 \\ 0 & 4 & d \end{bmatrix}.$$

7. (15pts) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Compute $A^T A$ and AA^T and their eigenvalues and unit eigenvectors.

(b) Find the singular values of A.

(c) Find a unit vector $\mathbf{x}$ so that $\|A\mathbf{x}\|$ is maximized. What is the maximum value?