Linear Algebra

Problem Set 10 **Spring 2016**

Due Tuesday, 7 June 2016 at 12:00 PM in EE105. This problem set covers Lecture 39-42. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) Let the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection about the line spanned by

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

   (a) Find an orthonormal eigenbasis $\boldsymbol{\beta}$ for T. Note that the basis $\boldsymbol{\beta}$ consists of orthonormal eigenvectors of T.
   (b) Find the matrix representation of T with respect to the basis $\boldsymbol{\beta}$.
   (c) Find the matrix representation of T with respect to the standard basis of $\mathbb{R}^3$.

2. (10pts) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

3. (10pts) Show that any symmetric positive definite matrix A can be written as $A = BB^T$, where B is a matrix with orthogonal columns. *Hint*: There is an orthogonal matrix Q such that $Q^{-1}AQ = D$ is a diagonal matrix with positive diagonal entries. Write D as the square of a diagonal matrix.

4. (20pts) True or false, with a reason or counterexample if false.
   (a) If A is a real symmetric matrix, then rank(A)=rank(A^2).
   (b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
   (c) If $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite, then a must be positive.
   (d) If A is an orthogonal matrix, then there must exist a symmetric invertible matrix S such that $S^{-1}AS$ is diagonal.
   (e) If A is symmetric and similar to B, then B is symmetric.
   (f) If A is any real matrix, then $A^T A$ is diagonalizable.
   (g) All symmetric positive definite matrices are invertible.
   (h) The inverse of a symmetric positive definite matrix is positive definite.

(i) The sum of two symmetric positive definite matrices is positive definite.

(j) The only symmetric positive definite projection matrix is $P = I$.

5. (20pts) Suppose A is real antisymmetric, i.e., $A^T = -A$.

   (a) Is A^2 antisymmetric as well? Or is A^2 symmetric? What can you say about the eigenvalues of A^2?

   (b) Show that $\mathbf{x}^T A\mathbf{x} = 0$ for every real vector $\mathbf{x}$.

   (c) Show that the eigenvalues of A are pure imaginary (including zero).

   (d) Show that the determinant of A is positive or zero.

6. (10pts) For what numbers c and d will A and B have positive eigenvalues? Test the 3 determinants:

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 & 0 \\ 3 & b & 4 \\ 0 & 4 & b \end{bmatrix}.$$

7. (15pts) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

   (a) Find the singular values of A.

   (b) Find orthonormal vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ such that $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$ are orthogonal.

   (c) Find a unit vector $\mathbf{x}$ so that $\|A\mathbf{x}\|$ is maximized. What is the maximum value?