Linear Algebra

Problem Set 2 **Spring 2016**

Due Tuesday, 15 March 2016 at 12:00 PM in EE105. This problem set covers Lectures 5-8. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (20pts) For two invertible $n \times n$ matrices A and B, determine which the following formulas are necessarily true.
   (a) A^2 is invertible, and $(A^2)^{-1} = (A^{-1})^2$
   (b) $A + B$ is invertible, and $(A+B)^{-1} = A^{-1} + B^{-1}$
   (c) $A^{-1}B$ is invertible, and $(A^{-1}B)^{-1} = B^{-1}A$
   (d) $(AB)^T$ is invertible, and $\left((AB)^T\right)^{-1} = \left(A^{-1}B^{-1}\right)^T$
   (e) $(A+B)^2 = A^2 + 2AB + B^2$
   (f) $(A+B)(A-B) = A^2 - B^2$
   (g) $ABA^{-1} = B$
   (h) $(ABA^{-1})^3 = AB^3A^{-1}$
   (i) $(I+A)(I-A) = I - A^2$
   (j) $ABB^{-1}A^{-1} = BB^{-1}A^{-1}A$
2. (20pts) Answer the following questions.
   (a) Can you find a 3×2 matrix A and a 2×3 matrix B such that the product AB is invertible? Explain.
   (b) If A is a singular $n \times n$ matrix, can you always find a nonzero $n \times n$ matrix B ($B \neq 0$) such that $AB = 0$? Explain.
   (c) Find a nonzero 2×2 matrix A such that $A^2 = 0$.
   (d) Find all invertible $n \times n$ matrices A such that $A^2 = A$.
3. (15pts) Assuming that the stated inverses exist, prove the following equalities and statements:
   (a) $(A+B)A^{-1}(A-B) = (A-B)A^{-1}(A+B)$
   (b) If $AB = A + B$, then $AB = BA$.
   (c) $(A^{-1}+B^{-1})^{-1} = A - A(A+B)^{-1}A$
4. (10pts) Solve the matrix equation for A:

$$A\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}.$$

   Do this using paper and pencil. Show all your work.

5. (15pts) It is known that $PA = LU$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 8 & -5 & 1 \end{bmatrix}, \ U = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 14 \\ -3 \\ 33 \\ 9 \end{bmatrix}$. Do this using paper and pencil. Show all your work.

(b) Find the first row of A^{-1}. Do this using paper and pencil. Show all your work.

6. (20pts) Answer the following questions.

(a) Consider two $n \times n$ matrices A and B whose entries a_{ij} and b_{ij} are positive or zero. Suppose that $a_{ij} \le p$, for $1 \le i, j \le n$, and all column sums of B are less than or equal to q, i.e., $\sum_{i=1}^{n} b_{ij} \le q$, for $1 \le j \le n$. Show that all entries of AB are less than or equal to pq.

(b) Let A be an $n \times n$ matrix whose entries are positive or zero. Suppose that all column sums of A are less than 1. Let s be the largest column sum of A. Show that the entries of A^m are less than or equal to s^m, for all positive integer m.

(c) From (b), show that $\lim_{m \to \infty} A^m = 0$ (meaning that all entries of A^m approach zero.