Linear Algebra

Problem Set 4 **Spring 2013**

Due Tuesday, 2 April 2013 at 12:00 PM in EE208. This problem set covers Lecture 11-16. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (10pts) Construct a matrix with the required property or use the rank-nullity theorem to explain why this is impossible:

   (a) Column space contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$.

   (b) Column space contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$.

   (c) Column space has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$, nullspace has basis $\begin{bmatrix}1\\1\end{bmatrix}$.

   (d) Row space has basis $\begin{bmatrix}1\\0\\0\end{bmatrix}$, nullspace has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

   (e) Column space = row space, nullspace = left null space.

2. (15pts) Let

$$A=\begin{bmatrix}1 & 4 & 6 & 2\\ 1 & 5 & 7 & 3\\ 1 & 3 & 5 & 1\end{bmatrix}.$$

   (a) Find the reduced row echelon form of A.

   (b) Find the nullspace matrix of A. Note that column space of the nullspace matrix equals the nullspace of A.

   (c) Find the complete solution to $A\mathbf{x}=\mathbf{0}$.

   (d) Determine the condition of $\mathbf{b}=(b_1,b_2,b_3)^T$ so that $A\mathbf{x}=\mathbf{b}$ is consistent (solvable).

   (e) Find the general solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}=(9,11,7)^T$.

3. (15pts) Consider the problem of solving linear system $A\mathbf{x}=\mathbf{b}$, where A is given below. Note that the dimension of $\mathbf{b}$ is determined by A.

$$(1)\ \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}\quad (2)\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\quad (3)\ \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\quad (4)\ \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\quad (5)\ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

(a) Which of the above system has at least one solution for every $\mathbf{b}$?
(b) Which of the above system has at most one solution for every $\mathbf{b}$?
(c) Which of the above system has exactly one solution for every $\mathbf{b}$?
(d) Which of the above system has infinite many solutions for every $\mathbf{b}$?
(e) Which of the above system has no solution for some $\mathbf{b}$?

4. (10pts) Suppose the columns of an m by n matrix A are linearly independent.
   (a) What is the rank of A?
   (b) What is the row space of A and the nullspace of A?
   (c) What is the relation of m and n?
   (d) Is $A\mathbf{x}=\mathbf{b}$ always solvable for every m-dimensional vector $\mathbf{b}$?
   (e) If $A\mathbf{x}=\mathbf{b}$ is solvable, is the solution necessarily unique?

5. (10pts) Suppose the general solution to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

What is A?

6. (15pts) The reduced row echelon form of $\begin{bmatrix} A & I_3 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 & 4 & -5 \\ 0 & 0 & 1 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (a) Find bases for the column space, row space, nullspace, and left nullspace of A.
   (b) Is $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ solvable? Why or why not.

(c) Is $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ solvable? Why or why not.

(d) What is the rank of the block matrix $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$? Think of its reduced row echelon form.

7. (10pts) True or false. If true, give your reasoning; otherwise, give a counterexample. All the following matrices are square.
   (a) If A and B have the same reduced row echelon form, then A and B have the same column space.
   (b) If A and B have the same reduced row echelon form, then A and B have the same row space.
   (c) The nullspace of A^2 equals the nullspace of A.
   (d) If A is invertible, then the nullspace of A^2 equals the nullspace of A.
   (e) If rank$A$=0, then A=0.
8. (15pts) Let A be an m by n matrix.
   (a) Let E be an m by m invertible matrix and B=EA. Prove that if

   $\mathbf{b}_j = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, then $\mathbf{a}_j = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$, where $\mathbf{a}_j$ and $\mathbf{b}_j$ denote

   the jth column of A and B, respectively, i.e., $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$ and

   $B = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$.

   (b) If E is not invertible, is the statement in (a) still correct? If yes, give a reason; if no, give a counter example.
   (c) From Problem 6, express $\mathbf{a}_2$ and $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3$, respectively.