Linear Algebra

Problem Set 4 **Spring 2015**

Due Tuesday, 7 April 2015 at 12:00 PM in EE106. This problem set covers Lecture 13-16. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (10pts) Construct a matrix with the required property or use the rank-nullity theorem to explain why this is impossible:

   (a) Column space contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

   (b) Column space contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$.

   (c) Column space has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$, nullspace has basis $\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}$.

   (d) Row space has basis $\begin{bmatrix}1\\0\\0\end{bmatrix}$, nullspace has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

   (e) Column space = row space, nullspace = left null space.

2. (15pts) Let

$$A=\begin{bmatrix}1 & 2 & 5 & 0\\1 & 3 & 7 & 1\\1 & 3 & 7 & 1\end{bmatrix}.$$

   (a) Find the reduced row echelon form of A.

   (b) Find the complete solution to $A\mathbf{x}=\mathbf{0}$.

   (c) Find the nullspace matrix of A. Note that column space of the nullspace matrix equals the nullspace of A.

   (d) Determine the condition of $\mathbf{b}=(b_1,b_2,b_3)^T$ so that $A\mathbf{x}=\mathbf{b}$ is consistent (solvable).

   (e) Find the general solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}=(8,11,11)^T$.

3. (10pts) Consider the problem of solving linear system $A\mathbf{x}=\mathbf{b}$, where A is given below. Note that the dimension of $\mathbf{b}$ is determined by A.

$$(1)\ \begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 4\end{bmatrix}\quad (2)\ \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\quad (3)\ \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}\quad (4)\ \begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix}\quad (5)\ \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$$

(a) Which of the above system has at least one solution for every **b**?
(b) Which of the above system has at most one solution for every **b**?
(c) Which of the above system has exactly one solution for every **b**?
(d) Which of the above system has infinite many solutions for every **b**?
(e) Which of the above system has no solution for some **b**?

4. (15pts) Suppose the rows of an m by n matrix A are linearly independent.
   (a) What is the rank of A?
   (b) What is the column space of A and the left nullspace of A?
   (c) What is the relation of m and n?
   (d) Is $A\mathbf{x}=\mathbf{b}$ always solvable for every m-dimensional vector $\mathbf{b}$?
   (e) If $A\mathbf{x}=\mathbf{b}$ is solvable, is the solution necessarily unique?
   (f) Can you always find a matrix B such that $AB=I_m$?
   (g) Can you always find a matrix B such that $BA=I_n$?

5. (10pts) Suppose the general solution to the equation $A\mathbf{x}=\begin{bmatrix}1\\2\\1\end{bmatrix}$ is

$$\mathbf{x}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+\alpha\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+\beta\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\gamma\begin{bmatrix}1\\1\\1\\0\end{bmatrix}.$$

   What is A?

6. (15pts) The reduced row echelon form of $\begin{bmatrix}A & I_3\end{bmatrix}$ is

$$\begin{bmatrix}1 & 2 & 0 & 1 & -1 & 5 & 2\\ 0 & 0 & 1 & 4 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 & 0 & 2 & 3\end{bmatrix}.$$

   (a) What is A?
   (b) Find bases for the column space, row space, nullspace, and left nullspace of A.

(c) Is $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ solvable? Why or why not.

(d) Is $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ solvable? Why or why not.

7. (15pts) Let A be an m by n matrix.
   (a) Let E be an m by m invertible matrix and B=EA. Prove that if

   $\mathbf{b}_j = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, then $\mathbf{a}_j = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$, where $\mathbf{a}_j$ and $\mathbf{b}_j$ denote

   the jth column of A and B, respectively, i.e., $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and

   $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$.
   (b) If E is not invertible, is the statement in (a) still correct? If yes, give a reason; if no, give a counter example.
   (c) From Problem 6, express $\mathbf{a}_2$ and $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3$, respectively.
8. (10pts) True or false. If true, give your reasoning; otherwise, give a counterexample. All the following matrices are square.
   (a) If A and B have the same reduced row echelon form, then A and B have the same column space.
   (b) If A and B have the same reduced row echelon form, then A and B have the same nullspace.
   (c) The nullspace of A^2 equals the nullspace of A.
   (d) If A is invertible, then the left nullspace of A equals the nullspace of A.
   (e) If rank(A^2)=0, then A=0.