Linear Algebra

Problem Set 4 **Spring 2016**

Due Tuesday, 29 March 2016 at 12:00 PM in EE105. This problem set covers Lectures 13-16. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) Consider the problem of solving linear system $A\mathbf{x}=\mathbf{b}$, where A is given below. Note that the dimension of $\mathbf{b}$ is determined by A.

$$(1)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\quad (2)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\quad (3)\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\quad (4)\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\quad (5)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

   (a) Which of the above system has at least one solution for every $\mathbf{b}$?
   (b) Which of the above system has at most one solution for every $\mathbf{b}$?
   (c) Which of the above system has exactly one solution for every $\mathbf{b}$?
   (d) Which of the above system has infinite many solutions for every $\mathbf{b}$?
   (e) Which of the above system has no solution for some $\mathbf{b}$?

2. (20pts) Let

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}.$$

   (a) Find a basis for the column space of A.
   (b) Find a basis for the row space of A.
   (c) Find a basis for the nullspace of A.
   (d) Find a basis for the nullspace of A^T.
   (e) Determine the condition of $\mathbf{b}=(b_1,b_2,b_3,b_4,b_5)^T$ so that $A\mathbf{x}=\mathbf{b}$ is solvable.
   (f) Find the general solution of $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}=(0,4,8,12,16)^T$.

3. (20pts) Suppose the columns of an m by n matrix A are linearly independent.
   (a) What is the rank of A?
   (b) What is the relation of m and n?
   (c) What is the reduced row echelon form of A?
   (d) What is the row space of A and the nullspace of A?
   (e) What is the dimension of the column space of A?
   (f) What is the dimension of the nullspace of A^T?
   (g) Is $A\mathbf{x}=\mathbf{b}$ always solvable for every m-dimensional vector $\mathbf{b}$?

(h) If $A\mathbf{x}=\mathbf{b}$ is consistent, is the solution necessarily unique?

(i) Can you always find a matrix B such that $AB=I_m$?

(j) Can you always find a matrix B such that $BA=I_n$?

4. (15pts) Consider a square matrix A.

   (a) What is the relationship between $N(A)$ and $N(A^2)$? Are they necessarily equal? More generally, what can you say about $N(A), N(A^2), N(A^3), \ldots$?

   (b) What can you say about $C(A), C(A^2), C(A^3), \ldots$?

   (c) If $N(A^2) = N(A^3)$, is it true that $N(A^3) = N(A^4)$? Justify your answer.

5. (5pts) Consider a p by n matrix A and a q by n matrix B, and form the partitioned matrix $C = \begin{bmatrix} A \\ B \end{bmatrix}$. What is the relationship between $N(A)$, $N(B)$, and $N(C)$?

6. (5pts) Consider an n by p matrix A and a p by m matrix B. You are told that the columns of A and the columns of B are linearly independent. Are the columns of AB linearly independent as well?

7. (5pts) Consider three linearly independent vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Are the vectors $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{v}+\mathbf{w}$ linearly independent as well?

8. (15pts) Consider linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in $\mathbb{R}^n$. Let

   $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_k \end{bmatrix}$. Let B be an invertible k by k matrix and C be an invertible n by n matrix.

   (a) Are the columns of AB linearly independent?

   (b) Are the columns of CA linearly independent?