Linear Algebra

Problem Set 5 **Spring 2012**

Due Tuesday, 10 April 2012 at 10:00 AM in EE208. This problem set covers Lecture 13-16. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$.
   (a) For any $\mathbf{x} \in \mathbb{R}^n$, prove that there is a unique way to express $\mathbf{x}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. (You may assume that there are two ways to do so, e.g., $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ and $\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_n\mathbf{v}_n$. Then show that $c_i = d_i$, for $i = 1, 2, \ldots, n$.)
   (b) Let A be an n by n invertible matrix. Prove that $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ is also a basis for $\mathbb{R}^n$.

   (Recall that all basis vectors are independent and they span $\mathbb{R}^n$.)
2. (15pts) Let A be an m by n matrix.
   (a) Let E be an m by m invertible matrix and $B=EA$. Prove that if $\mathbf{b}_j = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, then $\mathbf{a}_j = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$, where $\mathbf{a}_j$ and $\mathbf{b}_j$ denote the jth column of A and B, respectively, i.e., $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$.
   (b) Let R be the reduced row echelon form of A. If $R = \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, express $\mathbf{a}_2$ and $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3$, respectively.
   (c) If E is not invertible, is the statement in (a) still correct? If yes, give a reason; if no, give a counter example.
3. (15pts) Let A be a 3 by 5 matrix. The reduced row echelon form of $[A \ \ I_3]$ is as follows:

$$\begin{bmatrix} 1 & -2 & 3 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Determine A. Find bases for the four fundamental subspaces of A.

4. (20pts)

   Find a basis for each of these subspaces of $\mathbb{R}^n$:

   (a) All vectors whose components are equal.

   (b) All vectors whose components add to zero.

   (c) All vectors that are perpendicular to $(1,1,1,0)$ and $(1,0,1,1)$.

   (d) All vectors of the form (a,b,c,d), where $a=b$ and $c=d$.

   (For each problem, try to translate the requirement into the column space or nullspace of a matrix.)

5. (20pts) Construct a matrix with the required property or explain why this is impossible:

   (a) Column space has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$, nullspace has basis $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

   (b) Column space contains $\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\2\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}$.

   (c) Dimension of column space = dimension of nullspace.

   (d) Left nullspace contains $\begin{bmatrix}1\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\1\end{bmatrix}$.

   (e) Row space = column space, but nullspace $\neq$ left nullspace.

6. (15pts) Suppose the general solution to the equation

$$A\mathbf{x}=\begin{bmatrix}2\\2\\2\\2\end{bmatrix} \quad \text{is} \quad \mathbf{x}=\begin{bmatrix}1\\1\\1\end{bmatrix}+\alpha\begin{bmatrix}1\\-1\\-1\end{bmatrix},$$

   and the general solution to the equation

$$A\mathbf{x}=\begin{bmatrix}2\\2\\1\\1\end{bmatrix} \quad \text{is} \quad \mathbf{x}=\begin{bmatrix}1\\1\\0\end{bmatrix}+\alpha\begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

   (a) Determine rank$A$ and dim$N(A)$.

   (b) Find a basis for the column space of A.

   (c) Find the reduced row echelon form of A.