Linear Algebra

Problem Set 5 **Spring 2013**

Due Tuesday, 23 April 2013 at 12:00 PM in EE208. This problem set covers Lecture 18-21. Free feel to work with others, but the final write-up should be entirely based on your own understanding. Be sure to print your name and student ID on your homework.

1. (15pts) Find the standard matrix for T, determine rank(T) and bases for the range and kernel (like the column space and nullspace) of T:

   (a) $T(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ -x_1 + 2x_2 \end{bmatrix}$ (b) $T(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ x_1 - x_2 \end{bmatrix}$ (c) $T(x_1, x_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. (15pts) Suppose $T: V \to W$ is a linear transformation. Let $n = \dim V$, $m = \dim W$ and $r = \text{rank}(T)$.
   (a) If T is one-to-one, what is the relationship between m, n, and r?
   (b) If T maps V onto W, what can you say about m, n, and r?
   (c) If T is one-to-one and maps onto W, what can you say about m, n, and r?
3. (20pts) Let $T: V \to W$ be a linear transformation.
   (a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in V and T is one-to-one, explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent.
   (b) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set in V, is it true that set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent?
   (c) Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in V, but $\{T(\mathbf{u}), T(\mathbf{v})\}$ is a linearly dependent set. Show that there exists a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.
   (d) If $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ are linearly independent sets in V, are you sure that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set?
4. (15pts)
   (a) Find the coordinate vector of $p(t) = 2 - t + t^2$ relative to the basis

   $B = \{p_1(t), p_2(t), p_3(t)\}$, where $p_1(t) = 1 + t$, $p_2(t) = 1 + t^2$, $p_3(t) = t + t^2$.

   (b) Use coordinate vectors (with respect to any legal basis) to test the linear independence of the following set of 2 by 2 matrices:

   $$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$
5. (20pts) Suppose T is a linear transformation, and

$$T(1,1)=\begin{bmatrix}1\\2\end{bmatrix},\ T(1,-1)=\begin{bmatrix}2\\3\end{bmatrix}.$$

(a) What is $T(5,-1)$?

(b) If $T(x,y)=\begin{bmatrix}3\\5\end{bmatrix}$, determine the values of *x* and *y*.

(c) Find the matrix representation of *T* with respect to the standard basis

$S=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$, i.e., $[T]_S$.

(d) Find the matrix representation of *T* with respect to the basis

$B=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$, i.e., $[T]_B$. Recall that the first column of $[T]_B$ is the coordinate vector of $T(1,1)$ relative to basis *B*.

(e) Find the matrix representation of *T* with respect to the basis

$C=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\}$, i.e., $[T]_C$.

6. (15pts) Let *T*(*X*)=*AX*, where $X=\begin{bmatrix}x_{11} & x_{12}\\x_{21} & x_{22}\end{bmatrix}$ and $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$. Note that *T* maps 2 by 2 matrices into 2 by 2 matrices.

(a) Find the matrix representation of *T* with respect to the standard basis

$S=\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$. That is, find $[T]_S$.

(b) Find the matrix representation of *T* with respect to the basis

$B=\left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix},\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix},\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right\}$. That is, find $[T]_B$.